University of Freiburg
Dept. of Computer Science
Prof. Dr. F. Kuhn
S. Faour, M. Fuchs, G. Schmid

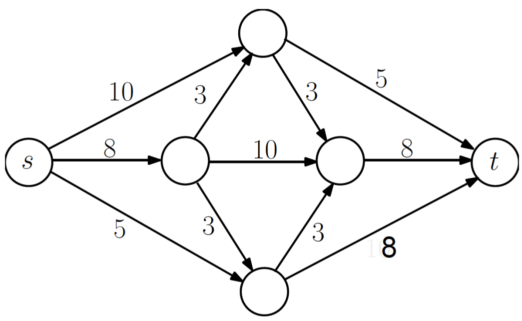
Algorithm Theory
Exercise Sheet 7

Due: Wednesday, 14th of December, 2022, 11:59 pm

Exercise 1: True or False? *(8 Points)*

Given a flow network $G = (V, E)$ with source s, sink t, and integer capacities. Verify whether the following statements are true or false and justify in both cases.

(a) Let G be the following flow network:



The maximum flow value of G is 21. *(3 Points)*
Hint: try applying the Ford Fulkerson algorithm.

(b) If all directed edges in G have distinct capacities, then there is a unique maximum flow. *(2 Points)*

(c) If we are also given a maximum flow of G, then increasing the capacity of an arbitrary edge $(u, v) \in E$ by one unit will also increase the new maximum flow value of the new network G' by one unit. *(3 Points)*

Exercise 2: Reductions to Max-Flow *(12 Points)*

(a) Given a flow network $G = (V, E)$ with source s, sink t, and integer capacities. Suppose that in addition to edge capacities, there are capacities at each node, i.e. a mapping $c : V \to \mathbb{N}$ denoted by $c(v)$.
Describe an algorithm that computes a maximum flow from the source to the sink such that it satisfies both the edge capacity constraints (and the conservation of flows) and the node capacity constraint i.e.

$$\forall v \in V \setminus \{s, t\} : \qquad \sum_{\{u \in V \mid (u,v) \in E\}} f_{uv} \leq c(v).$$

(5 Points)

(b) Given a directed graph $G = (V, E)$ where each directed edge e has an integer capacity $c_e \geq 0$. In this problem setup there is no designated source nor sink, but instead we have nodes with a supply and nodes with a demand, i.e. a node can have a supply of e.g. 10 units, which means it can send out 10 more units flow than it receives; and a node can have a demand of e.g. 10 units, which means it wants to receive 10 more units flow than it sends out.

To formalize this idea, we first define the *excess* of a node v with respect to a feasible flow f to be the difference between the amount of flow entering it and the amount of flow leaving it i.e $f^{in}(v) - f^{out}(v)$.

Next, we associate for each node $v \in V$ an integer value d_v that represents the demand of node v. If $d_v < 0$, then it represents its supply. Otherwise, v doesn't have a supply nor demand.

Describe an algorithm that solves the following decision problem: is there a flow that satisfies the edge capacity constraints *and* demand constraints i.e.

$$\forall v \in V, f^{in}(v) - f^{out}(v) = |d_v|?$$

Argue correctness and analyze its running time. *(7 Points)*

Remark: you can assume that the total supply and demand in the network are the same.