University of Freiburg
Dept. of Computer Science
Prof. Dr. F. Kuhn
S. Faour, M. Fuchs, G. Schmid

Algorithm Theory
Exercise Sheet 11

Due: Wednesday, 25th of January, 2023, 11:59 pm

Exercise 1: Max Cut *(8 Points)*

Let $G = (V, E)$ be a simple undirected graph. Consider the following randomized algorithm: Every node $v \in V$ joins set S with probability $1/2$. You can assume that $(S, V \setminus S)$ actually forms a cut i.e. $\emptyset \neq S \neq V$.

(a) Show that with probability at least $1/3$ this algorithm outputs a cut which is a 4-approximation to the maximum cut (i.e., the cut of maximum possible size) *(5 Points)*
Hint: Apply the Markov inequality to the number of edges that do ***not*** *cross the cut. For a non-negative random variable X,the Markov inequality states that for all $t > 0$ we have*

$$P(X \geq t) \leq \frac{E[X]}{t}$$

(b) How can you use the above's algorithm to devise a 4-approximation with probability at least $1 - \left(\frac{2}{3}\right)^k$ for any integer $k > 0$? *(3 Points)*

Exercise 2: Randomized Coloring *(12 Points)*

Let $G = (V, E)$ be a simple, undirected graph with maximum degree Δ. A (node) coloring of the graph is an assignment of colors to the nodes in a way that no two adjacent nodes are assigned with the same color. More formal: A coloring is a mapping $\phi : V \to C$ of nodes in V to some color space C s.t. $\phi(u) \neq \phi(v)$ if $\{u, v\} \in E$.
Consider the following algorithm to assign colors from the colors pace $C = \{1, 2, \ldots, \Delta + 1\}$ to the nodes. Let L_v be the lists of **available** colors of v, that initially is set to $L_v := C$.

Algorithm 1 Randomized Coloring

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Ensure: φ is a proper Δ + 1 coloring
Let L_v := {1, 2, ..., Δ + 1}
for each uncolored node v ∈ V in parallel do
    v becomes active with probability p = 1/2
    if v is active then
        Let v choose a color x_v ∈ L_v uniformly at random  ▷ same probability for each color in L_v
        if no neighbor u picked x_v as well then
            φ(v) := x_v                                     ▷ v is colored now!
    Delete colors of neighbors from L_v
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Note that in every iteration, the size of L_v of an uncolored node v is larger than the number of uncolored neighbors of v.

(a) Show that a node v that is still uncolored will be colored in the next iteration with probability at least $1/4$. *(6 Points)*
Hint: Assume v is active and has k uncolored neighbors. What is the probability that v gets colored?

(b) After how many iterations is a node $v \in V$ colored in expectation? *(2 Points)*

(c) Show that Algorithm 1 terminates in $O(\log n)$ iterations **with high probability**.
That is for a given constant $c > 0$, all nodes are colored within $O(\log n)$ iterations with probability at least $1 - \frac{1}{n^c}$. *(4 Points)*
Hint: Use the result of a) for tasks b) and c) even if you didn't manage to come up with a solution.