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Dept. of Computer Science
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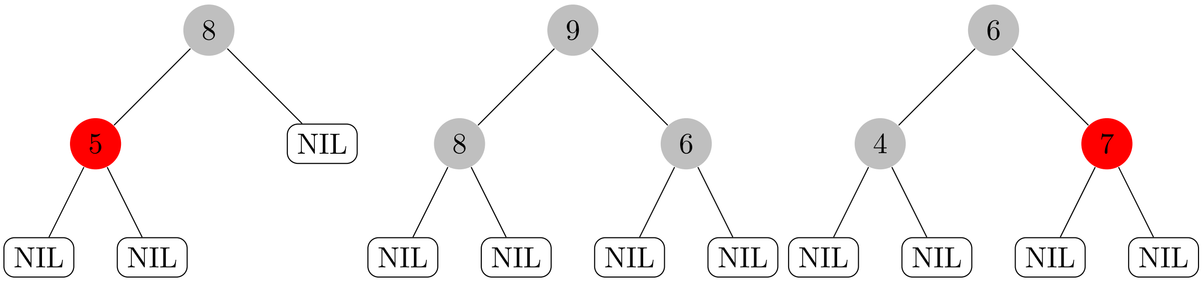
Algorithms and Datastructures
Winter Term 2023
Sample Solution Exercise Sheet 6

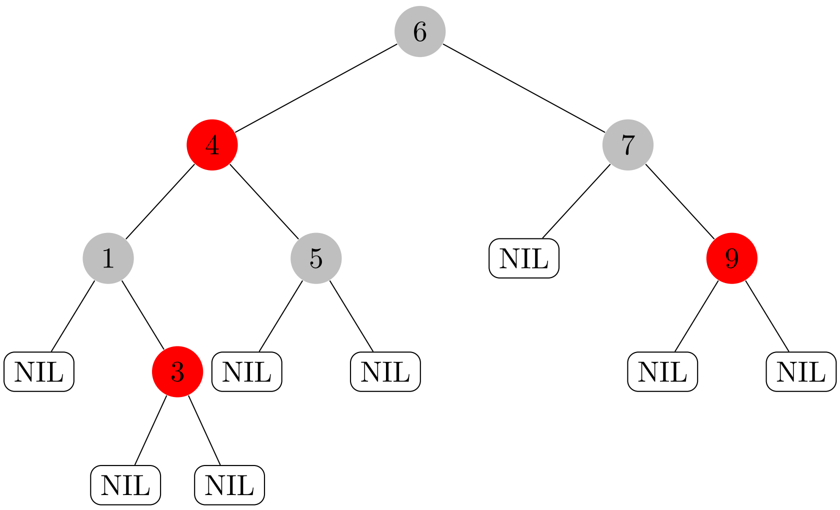
Due: Wednesday, December 13th, 2pm

Exercise 1: Red-Black Trees *(10 Points)*

(a) Decide for each of the following trees if it is a red-black tree and if not, which property is violated:



(b) On the following red-black tree, first execute the operation `insert`(8) and afterwards `delete`(5). Draw the resulting tree and document intermediate steps.

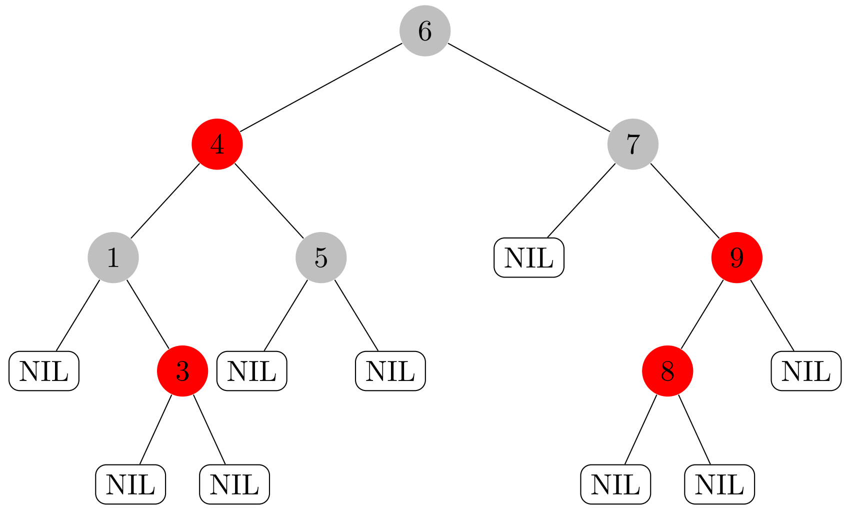


Sample Solution

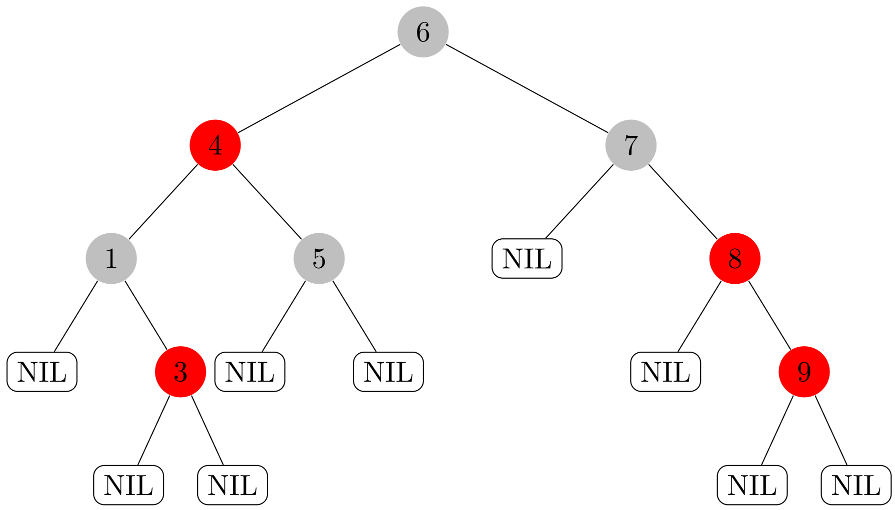
(a) From left to right:

1) Red-black-tree
2) No red-black-tree, because it is no binary search tree (the root's right child has a smaller key).
3) No red-black-tree, because the number of black nodes on a path from the root to a leaf is larger if you go through the left subtree.

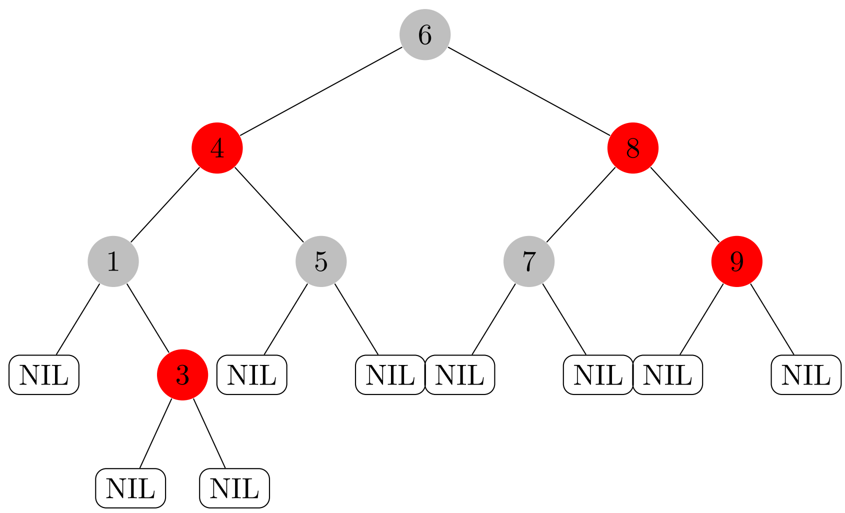
(b) We insert a red node with key 8 according to the rule of inserting into binary search trees.



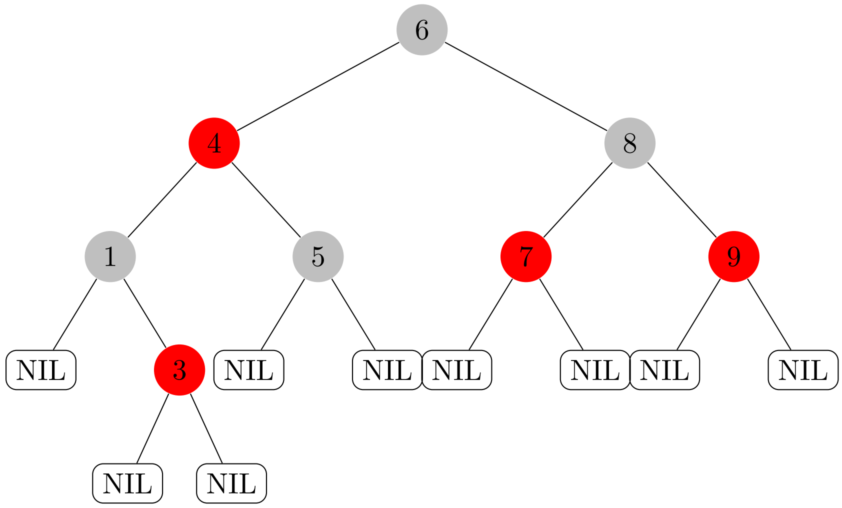
We are in case 1b from the lecture. We do a right-rotate(9,8),



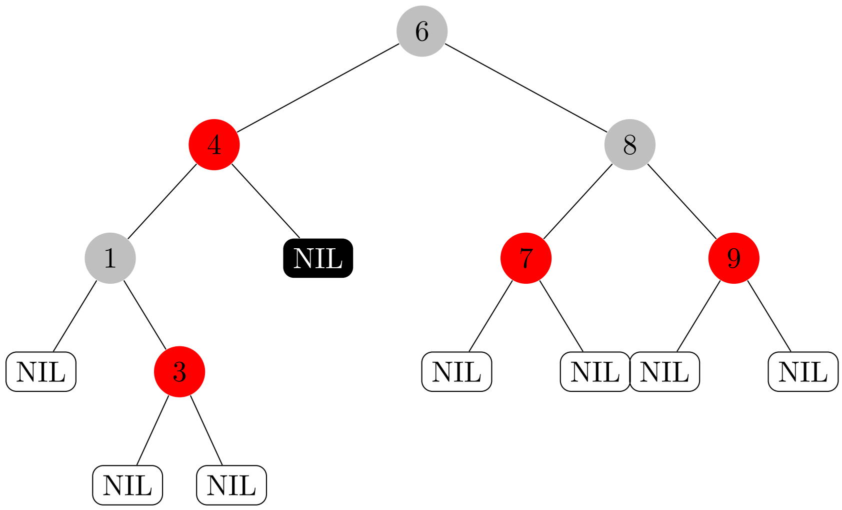
a left-rotate(7,8)



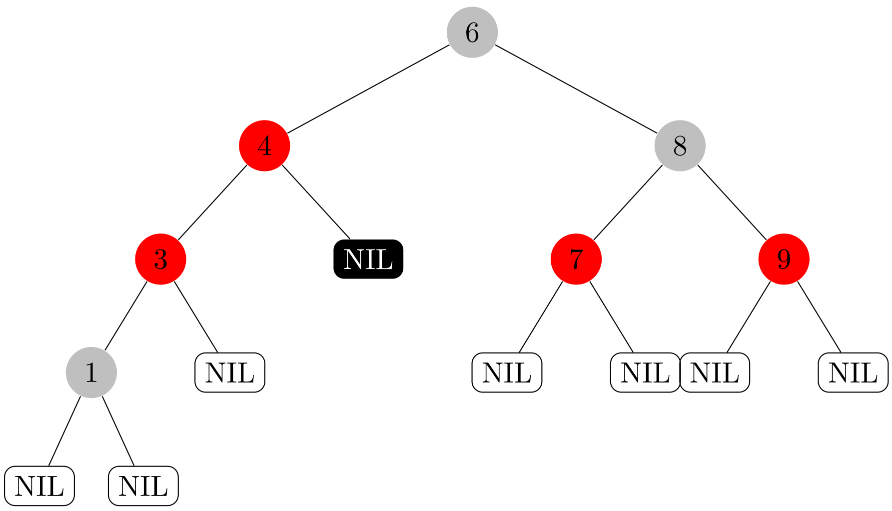
and recolor nodes 7 and 8.



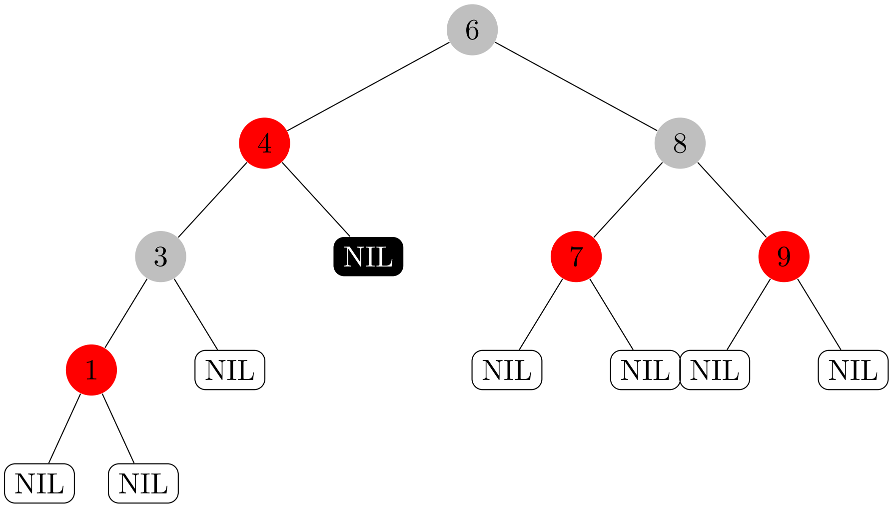
Now we execute `delete`(5). We are in case 2b from the lecture (deleting a black node with two NIL-children). First we remove node 5 from the tree and color the right NIL-child of node 4 double black to correct the black height.



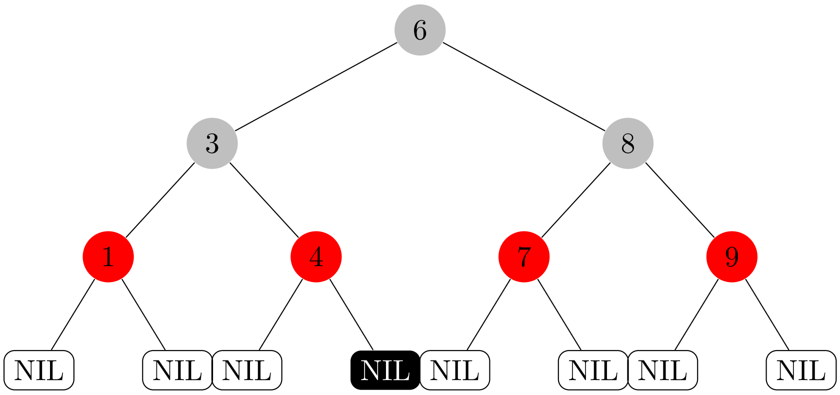
We are in case A.2 from the lecture. We do a left-rotate(1,3)



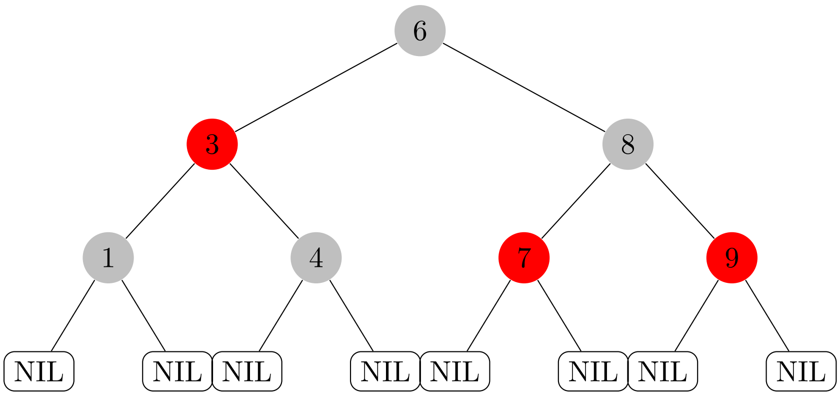
and recolor nodes 1 and 3.



Now we are in case A.1. We do a right-rotate(4,3)



and recolor. Finally, the tree looks like this.



Exercise 2: AVL-Trees [1] *(10 Points)*

An AVL-tree is a binary search tree with the additional property that for each node v, the depth of its left and its right subtree differ by at most 1.

(a) Show via induction that an AVL-tree of depth d is filled completely up to depth $\lfloor \frac{d}{2} \rfloor$. *(3 Points)*
A binary tree is filled completely up to depth d' if it contains for all $x \leq d'$ exactly 2^x nodes of depth x.

[1] AVL-Trees are not part of the lecture. To solve this exercise the definition given below is sufficient.

(b) Give a recursion relation that describes the minimum number of nodes of an AVL-tree as a function of d. *(3 Points)*

(c) Show that an AVL-tree with n nodes has depth $\mathcal{O}(\log n)$. *(4 Points)*

You can either use part (a) or part (b).

Sample Solution

(a) **Induktion start:** Each non-empty tree has a root and is hence completely filled up to depth 0. Hence the statement is true for $d = 0$ and $d = 1$ (as $\lfloor d/2 \rfloor = 0$ for $d = 0$ and $d = 1$).

Induktion step: Assume the statement holds for all AVL-trees up to depth d. We show that it also holds for AVL-trees of depth $d + 1$.

Let T be an AVL-tree of depth $d + 1$ with r as root and T_ℓ and T_r as left and right subtree. One of these subtrees must have depth d (lets say T_ℓ). As T is an AVL-tree, it follows that T_r has depth at least $d - 1$. By the induction hypothesis, T_ℓ is completely filled up to depth $\lfloor d/2 \rfloor$ and T_r is completely filled up to depth $\lfloor \frac{d-1}{2} \rfloor$. So both subtrees are completely filled up to depth $\lfloor \frac{d-1}{2} \rfloor = \lfloor \frac{d+1}{2} - 1 \rfloor = \lfloor \frac{d+1}{2} \rfloor - 1$ and hence T is filled completely up to depth $\lfloor \frac{d+1}{2} \rfloor$.

(b) Let n_d be the minimum number of nodes in an AVL-tree of depth d. As every tree of depth d has at least $d + 1$ nodes (as it contains a path of length d), we obtain as base cases $n_0 = 1$ and $n_1 = 2$. Now let $d \geq 2$. An AVL-tree T of depth d consists of a root r, a left subtree T_ℓ and a right subtree T_r. One of them, lets say T_ℓ, has depth $d - 1$ and hence at least n_{d-1} nodes. As T is an AVL-tree, it follows that T_r has depth at least $d - 2$ and hence at least n_{d-2} nodes. Hence T has at least $n_d = n_{d-1} + n_{d-2} + 1$ nodes.

(c) **Using (a)**: And AVL-tree of depth d is filled completely up to depth $\lfloor \frac{d}{2} \rfloor$, so T has $n \geq 2^{\lfloor \frac{d}{2} \rfloor}$ nodes. We obtain

$$\begin{aligned}
& 2^{\lfloor \frac{d}{2} \rfloor} \leq n \\
\iff\ & \lfloor \frac{d}{2} \rfloor \leq \log(n) \\
\implies\ & \frac{d}{2} - \frac{1}{2} \leq \lfloor \frac{d}{2} \rfloor \leq \log(n) \\
\implies\ & d \leq 2 \log n + 1 \\
\implies\ & d \in \mathcal{O}(\log(n)).
\end{aligned}$$

Using (b): Similar to the Fibonacci-series we have $n_d = n_{d-1} + n_{d-2} + 1 = 2n_{d-2} + n_{d-3} + 2 \geq 2n_{d-2}$. This means that increasing the depth by 2 doubles the number of nodes, so the number of nodes grows exponentially in the depth, or the depth grows logarithmically in the number of nodes. More formally, we have $n_d \geq 2n_{d-2} \geq 2^2 n_{d-4} \geq \cdots \geq 2^{\lfloor d/2 \rfloor} n_{d-2\lfloor d/2 \rfloor} \geq 2^{\lfloor d/2 \rfloor} n_0 = 2^{\lfloor d/2 \rfloor}$. The rest follows as above.