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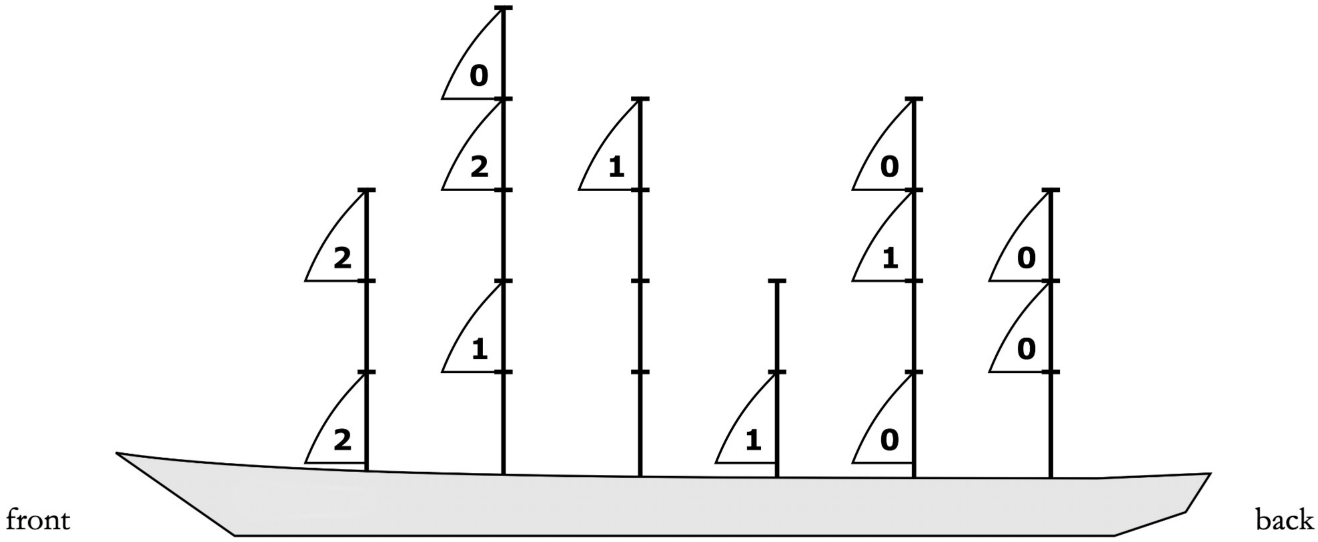
Algorithm Theory
Exercise Sheet 2

Due: Friday, 10rd of November 2023, 10:00 am

Exercise 1: Migrating Pirates *(10 Points)*

The great pirates of the seven seas have decided to travel to unknown seas. For this purpose they have decided to create a new ship with N masts (poles). Every mast is divided into unit sized segments - the height of a mast is equal to the number of its segments. Each mast is fitted with a number of sails (flags) and each sail exactly fits into one segment. Given a distribution of the sails on the masts you can calculate the total inefficiency of this configuration. Which is calculated in the following way. For every flag you calculate how many flags are behind (right to) it at the same height, this gives you the inefficiency for a given sail. You add up all of these individual inefficiencies, and you get the total inefficiency. For the problem we are given the following numbers: N pairs of numbers (a_i, b_i), where a_i is the height of the ith mast and b_i $(a_i \geq b_i)$ is the number of sails on the ith mast. Your task is to distribute the sails on the masts, so the total inefficiency is minimized.



This ship has 6 masts, of heights 3, 5, 4, 2, 4 and 3 from front (left side of image) to back.
This distribution of sails gives a total inefficiency of 10. The individual inefficiency of each sail is written inside the sail.

(a) Solve the problem if every mast hast the same height, in other words $\forall i, j$ $a_i = a_j = C$, where C is a constant.
Hint: Look at the levels of the mast. How should the flags be distributed? *(10 Points)*

(b) *Bonus Points Problem:* Find a greedy algorithm for the original problem and prove that it is optimal. (Currently we do not know the proof.) *(10 Points)*

Exercise 2: More about the Matroid Greedy algorithm *(10 Points)*

In the following problems, we assume that you are given a matroid $M = (S, F)$, a cost function $c : S \to \mathbb{R}_+$, and an independence oracle, that is You have a black box function that gives you back whether a set $A \subset S$ is independent or not. Your algorithms should run in polynomial time in $|S|$ and the number of Oracle calls. Additionally, you can use the following definition and theorem in your proofs.

Definition 0.1. We call a maximal independent set in M a basis or a base of M.

Definition 0.2. A minimal dependent set in an arbitrary matroid $M = (S, F)$ will be called a circuit(cycle) of M. Here a minimal dependent set means if you take away any element from it, it becomes independent. (If you have a graph you would call this a minimal cycle.)

Theorem 1. *If you are given a base B of the matroid, and you add an element $e \notin B$ to B, there will be a unique cycle in $B \cup e$ which contains e. If you remove any element of this cycle from $B \cup e$ you get a base of the matroid.*

(a) Prove that if every value of $c(s), s \in S$ is unique, then you have a unique maximum weight base. *(3 Points)*

(b) Given two cost functions for the base elements of the matroid c_1, c_2. Find a base that has the maximum weight according to c_1, and among these have the maximum weight according to c_2. You also need to prove correctness. *(4 Points)*

(c) Use the previous algorithm to algorithmically solve the following problem. Given an independent set G decide if it can be extended to a maximum weight base. You also need to prove correctness. *(3 Points)*