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Algorithm Theory
Exercise Sheet 7

Due: Friday, 8th of December, 2023, 10:00 am

Exercise 1: Fibonacci Heap - Amortized *(6 Points)*

Suppose we "simplify" Fibonacci heaps such that we do *not* mark any nodes that have lost a child and consequentially also do *not* cut marked parents of a node that needs to be cut out due to a `decrease-key`-operation. Is the *amortized* running time

(a) ... of the `decrease-key`-operation still $\mathcal{O}(1)$? *(2 Points)*

(b) ... of the `delete-min`-operation still $\mathcal{O}(\log n)$? *(4 Points)*

Explain your answers.

Exercise 2: Cuts and Flows *(14 Points)*

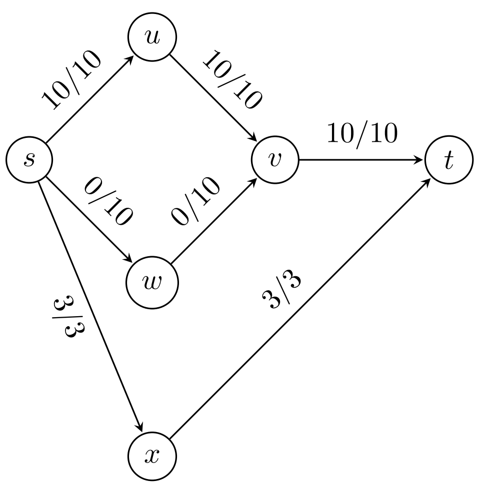
Note that the following three tasks are independent of each other.

(a) Given a flow-network G with nonnegative integer capacities on edges, we call an edge $e \in E$ saturated if its flow value equals its capacity, i.e., if $f(e) = c_e$. Prove or disprove the following statements.

  (1) If an edge e is crossing a minimum s-t-cut of G, any execution of the Ford-Fulkerson algorithm will saturate e. *(2 Points)*

  (2) Given some maximum flow of G that saturates edge e, then e is crossing at least one minimum s-t cut of G. *(2 Points)*

(b) Let the figure below represent a flow-network G with positive integer capacities on edges and a maximum flow f^*, where we denote the flow value f^* and the capacity c by f^*/c on the corresponding edge.



In the lecture we have seen that given a maximum flow, one can compute a minimum s-t cut $(A^*, V \setminus A^*)$, where A^* is the set of nodes that can be reached from s on a path with positive residual capacities in the residual graph. Give the cut $(A^*, V \setminus A^*)$ in G. *(3 Points)*

(c) Given a flow-network G with a source s, sink t, and nonnegative integer capacities on edges.

(1) Consider a minimum s-t cut $(S, V \setminus S)$ of G. Prove that $(S, V \setminus S)$ is not unique *if and only if* there exists an edge e crossing the cut $(S, V \setminus S)$ such that after increasing the capacity of e by 1, the capacity of the new minimum s-t cut is the same as the capacity of the old minimum s-t cut. *(5 Points)*
Remark: A minimum s-t cut $(S, V \setminus S)$ in G is said to be unique if and only if the capacity of the cut $(S, V \setminus S)$ is strictly less than the capacity of any other s-t cut $(F, V \setminus F)$ in G.

(2) Give a polynomial-time algorithm to decide whether G has a unique minimum s-t cut or not. *(2 Points)*