



Algorithm Theory

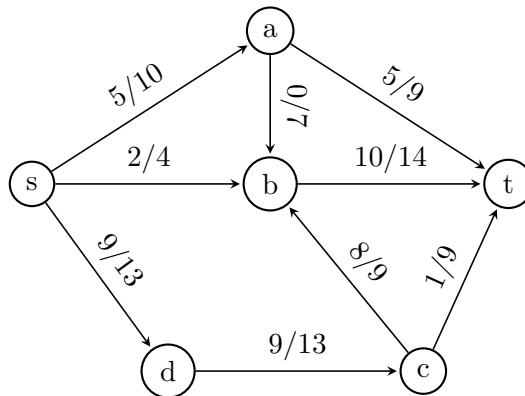
Sample Solution Exercise Sheet 8

Due: Friday, 15th of December 2023, 10:00 am

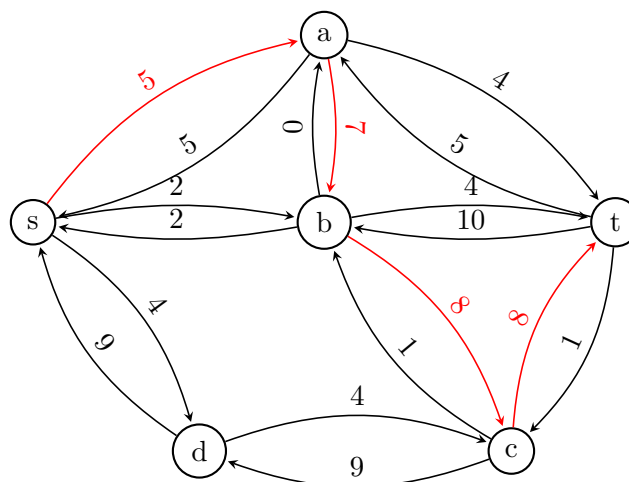
Exercise 1: Ford-Fulkerson Algorithm

(6 Points)

Consider the following flow network, where for each edge, the capacity (second number) and a current flow value (first number) are given. Solve the maximum flow problem on the network by using the Ford-Fulkerson variant that always picks a best possible augmenting path (an augmenting path that improves the current flow from s to t by as much as possible) in every iteration. Give intermediate results, i.e., draw the residual graph with all the residual capacities in every iteration.

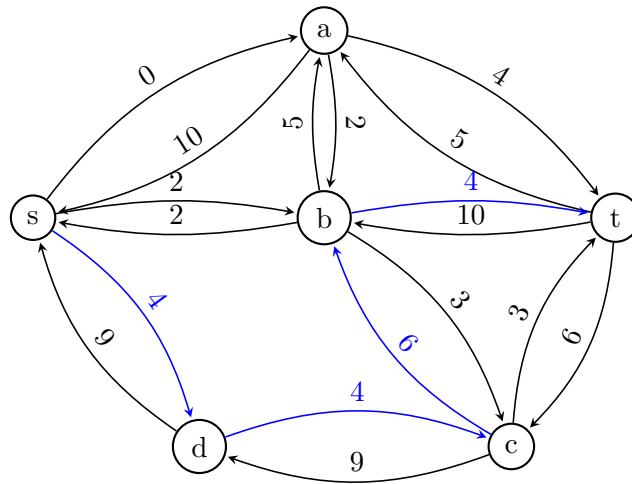


Sample Solution



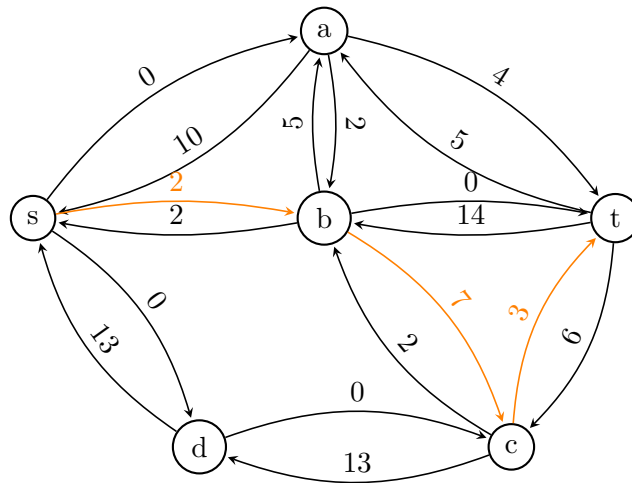
The best possible augmenting path is: $s \rightarrow a \rightarrow b \rightarrow c \rightarrow t$. This path increases the flow by 5. Hence, the resulting flow value is 21.

The best possible augmenting path now is: $s \rightarrow d \rightarrow c \rightarrow b \rightarrow t$. This path increases the flow by 4.



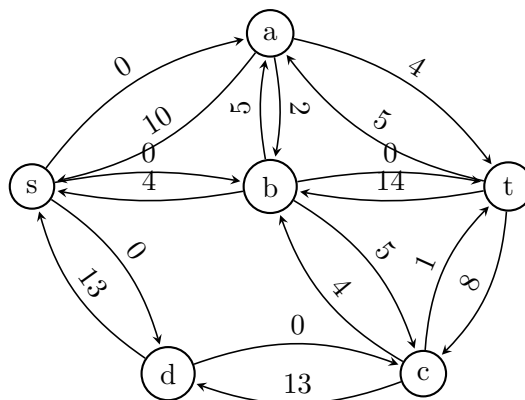
Hence, the resulting flow value is 25.

The best possible augmenting path now is: $s \rightarrow b \rightarrow c \rightarrow t$. This path increases the flow by 2. Hence,



the resulting flow value is 27.

No further improvement possible. Thus, the max. flow is 27.



Exercise 2: Seating Arrangement

(4 Points)

A group of students goes out to eat dinner together. To increase social interaction, they would like to sit at tables such that no two students from the same faculty are at the same table. For that purpose assume there are students from x different faculties while n_1, n_2, \dots, n_x describe the affiliated number

of students from these x faculties. Also assume that there are y tables available while r_j students can take place on the j -th table.

Let us define the *seating arrangement* as the decision problem returning **true** if one can distribute the students from same faculties to different tables and **false** otherwise. Formulate this *seating arrangement* problem as a *maximum flow* problem and write down the condition that should hold whenever the original decision problem returns **true**. Further, give the runtime it takes to solve the corresponding flow problem in terms of x, y, n_i and r_j for all $1 \leq i \leq x$ and $1 \leq j \leq y$.

Sample Solution

The idea is to construct a complete bipartite graph G , where the left side consists of the faculties and the right side consists of the possible tables. Each edge in between has capacity 1. Adding a node s on the left, connected to all faculties with corresponding n_i capacities and a node t connected to all tables with corresponding r_i capacities completes the graph. The seating arrangement problem will return **true**, when the maximum flow of G is $\sum_{i=1}^x n_i$ and **false** otherwise. The runtime of Ford-Fulkerson is $O(m \cdot C)$ where $m = x + y + x \cdot y = O(x \cdot y)$ and the maximum flow C is at most $\min \{ \sum_{i=1}^x n_i, \sum_{i=1}^y r_i \}$. (Note that one can refine the runtime by the fact that if $\sum_{i=1}^y r_i < \sum_{i=1}^x n_i$ there is no proper seat arrangement possible.)

Alternatively, a more formal definition of the graph:

$$F = \{F_1, F_2, \dots, F_x\}$$

$$T = \{T_1, T_2, \dots, T_y\}$$

$$G = (V, E) \text{ with}$$

$$V = \{s, t\} \cup F \cup T$$

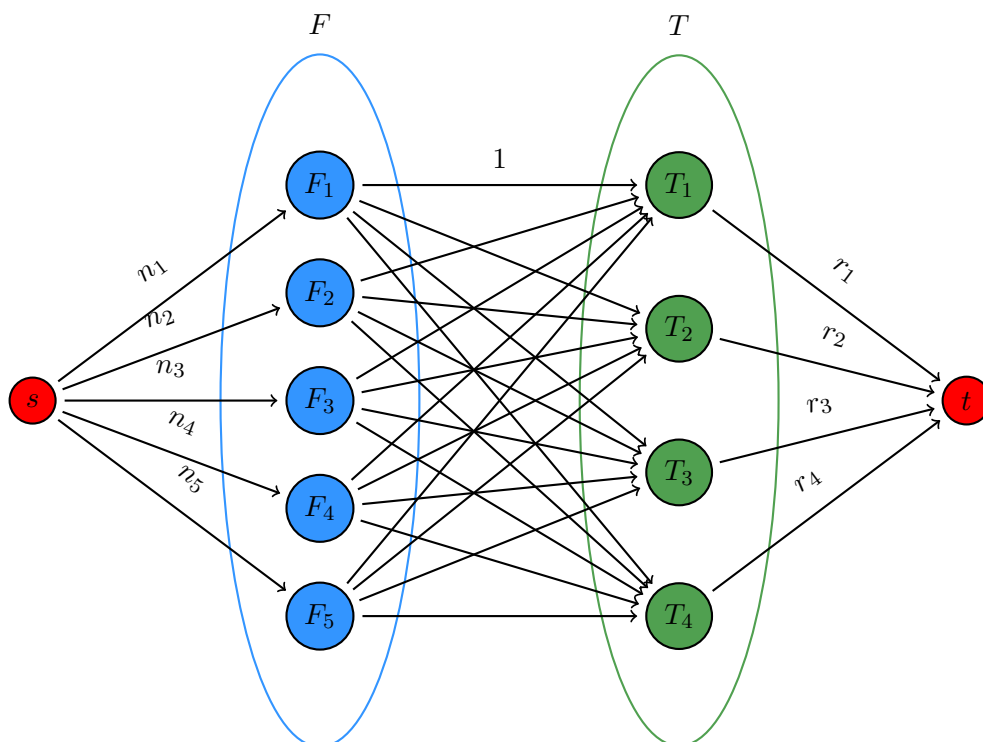
$$E = \{(s, v) | v \in F\} \cup \{(u, v) | u \in F, v \in T\} \cup \{(v, t) | v \in T\}$$

where the capacities are like follows:

$$c(s, v) = n_i \text{ if } v = F_i$$

$$c(u, v) = 1 \text{ if } u \in F, v \in T$$

$$c(v, t) = r_i \text{ if } v = T_i$$



Exercise 3: Smallest Minimum Cut

(10 Points)

Let $G = (V, E)$ be a flow network with *integer capacities* $c_e \geq 0$ for all $e \in E$. Give a new flow network $G' = (V, E)$ (that has the same nodes and edges as G) with *integer capacities* $c'_e \geq 0$ such that any minimum cut in G' is a minimum cut in G with the smallest number of edges (of all minimum cuts in G). Proof your statement!

Hint: Consider capacities $c'_e := c_e + 1$. How does the capacity of a cut change by this choice? Does this already solve the task or do you need to adjust it?

Sample Solution

Let $m := |E|$ and $n := |V|$. The intuition is the following: If we would set $c'_e = c_e + 1$, we increase the capacity of each cut (compared to the original value) by the number of edges of the cut. However, the problem is that cuts that were not minimal in the first place, but contain just few edges, may become minimum cuts by that choice of c'_e . To overcome this problem we use that no cut is containing more than m edges.

We define the new capacity for each edge e like follows: $c'_e := m \cdot c_e + 1$. Let in the following $(C, V \setminus C)$ be a minimum cut in G' . Now we want to show that $(C, V \setminus C)$ is also a min cut in G and second, that $(C, V \setminus C)$ has the smallest number of edges among the min cuts of G . We will now show the first of these statements by contradiction i.e., assume there is a cut $(S, V \setminus S)$ that has a smaller capacity than $(C, V \setminus C)$ in G . We start measuring the capacity of this cut in G' and use that the capacity of $(S, V \setminus S)$ is at least 1 unit smaller than the capacity of $(C, V \setminus C)$ in G since we have integer capacities.

$$\begin{aligned} \sum_{e \in (S, V \setminus S)} c'_e &= \sum_{e \in (S, V \setminus S)} (m \cdot c_e + 1) \\ &= m \sum_{e \in (S, V \setminus S)} c_e + \sum_{e \in (S, V \setminus S)} 1 \\ &\leq m \left(-1 + \sum_{e \in (C, V \setminus C)} c_e \right) + m \\ &= \sum_{e \in (C, V \setminus C)} m \cdot c_e \\ &< \sum_{e \in (C, V \setminus C)} (m \cdot c_e + 1) \\ &= \sum_{e \in (C, V \setminus C)} c'_e \end{aligned}$$

Thus, the capacity of $(S, V \setminus S)$ would be smaller than the capacity of $(C, V \setminus C)$ in G' and that is a contradiction the definition of $(C, V \setminus C)$. Now we know that $(C, V \setminus C)$ is also a minimum cut in G . It remains to show that we use a minimum number of edges in G . Again, assume for contradiction purpose that there is a cut $(S', V \setminus S')$, that is also a minimum cut in G but uses fewer edges i.e. $\sum_{e \in (S', V \setminus S')} 1 < \sum_{e \in (C, V \setminus C)} 1$ but $\sum_{e \in (S', V \setminus S')} c_e = \sum_{e \in (C, V \setminus C)} c_e$. Contradiction follows

by the following lines.

$$\begin{aligned}\sum_{e \in (S', V \setminus S')} c'_e &= \sum_{e \in (S', V \setminus S')} (m \cdot c_e + 1) \\ &= m \sum_{e \in (S', V \setminus S')} c_e + \sum_{e \in (S', V \setminus S')} 1 \\ &< m \sum_{e \in (C, V \setminus C)} c_e + \sum_{e \in (C, V \setminus C)} 1 \\ &= \sum_{e \in (C, V \setminus C)} (m \cdot c_e + 1) \\ &= \sum_{e \in (C, V \setminus C)} c'_e\end{aligned}$$