University of Freiburg
Dept. of Computer Science
Prof. Dr. F. Kuhn
S. Faour, M. Fuchs, A. Malyusz

Algorithm Theory
Sample Solution Exercise Sheet 9

Due: Friday, 22nd of December 2023, 10:00 am

Exercise 1: Mining Operations *(12 Points)*

The FY Corporation has decided to begin mining operations on a remote island. They have done preliminary tests, so they know what kind of jobs (operations) they can do. They know that there are n operations available, all of them has p_i, $\forall i = 1, \ldots, n$ value (this can be negative). They also know that some operations are prerequisites for other operations, e.g., the job i has to be completed before j. It can also happen that an operation has many prerequisites. Your task is to find a set of jobs S that are prerequisite complete, meaning every operation includes every other operation that is a prerequisite for it, in the set S, such that the sum of the p_i for these jobs is maximum. Give a polynomial-time algorithm that achieves this solution.

Sample Solution

Consider the following graph D. Let the vertex set be the following: For every operation there is a vertex representing it (we simply use the index of the operation to talk about the vertex), additionally there is an s and t vertices to help create a flow. Now let us define the directed edges of this graph. If p_i is positive let there be a directed edge from s to i with capacity p_i. If p_i is negative let there be a directed edge from i to t with capacity $-p_i$. If i operation is a prerequisite for the j operation then there is a directed edge from j to i with ∞ capacity. With the help of orientation of this edge we can achieve that in a min cut no such an edge lies in it. Now look at a min cut in this directed graph. The operations in the same cluster as s are the jobs we will do and the jobs in the t cluster are the ones we will not. No edge with infinite capacity will cross the min cut. Proof: There exists a cut where this holds as such for a min cut this also holds. Because of this the operations in the s cluster are prerequisite complete. The capacity of the cut $c(S,T) = \sum_{i \in T: p_i > 0} p_i + \sum_{j \in S: p_j < 0} (-p_j) = \sum_{i: p_i > 0} p_i - \sum_{i \in S: p_i} p_i$. Here the first term is constant and the second term being minimized ($-$profit) is equivalent to maximizing profit.

Exercise 2: Matching in Bipartite Graphs *(2+2+4 Points)*

(a) Prove that in a k-regular (every vertex has degree k) bipartite graph there exists a perfect matching.

(b) Prove that in a k-regular bipartite graph the edge set can be partitioned into k perfect matchings.

(c) Prove that in a bipartite graph there exists a matching which covers every vertex that has maximum degree.

Sample Solution

Let the graph be $G = (A \cup B, E)$

(a) Use Hall's theorem: For every $U \subset A$ the $N(U)$ set have size at least $|U|$ because we have a k-regular graph (Count the number of edges in two way.)

(b) Use repeatedly (a)

(c) Add to the graph new edges and vertices until the graph is δ regular then, because of (a) we have a perfect matching. The originally maximum degree nodes did not get any new edge as such if we delete the newly added nodes and edges we get a matching covering these nodes.