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Algorithms and Datastructures
Winter Term 2022
Exercise Sheet 10

Due: Wednesday, January 22nd, 2pm

Exercise 1: Dijkstra's Algorithm *(10 Points)*

In the lecture we saw that the runtime of Dijkstra using Fibonacci heaps is $O(m + n \log n)$. Is this the actual runtime of the algorithm? Maybe our analysis is just not good enough! We will show that the analysis is indeed tight.

(a) Argue that any algorithm solving SSSP (Single Source Shortest Paths) must spend at least $\Omega(m)$ time. An intuitive explanation is sufficient. *(1 Point)*

(b) Proof that the Dijkstra algorithm determines shortest paths in a sorted order. For a source node v and any other two nodes $u \neq w$ then u will be marked before w if $d(v,u) < d(v,w)$. Give a formal proof. *(4 Points)*

(c) Proof that Dijkstras Algorithm needs $\Omega(n \log n)$ time if the algorithm is implemented using a comparison based heap. The idea is the following: reduce the problem of sorting n numbers to the SSSP problem. (Given n numbers in an array A create an instance of SSSP.) Give a precise description and a formal proof. *(5 Points)*

Exercise 2: Currency Exchange *(10 Points)*

Consider n currencies $w_1, \dots, w_n$. The exchange rates are given in an $n \times n$-matrix A with entries a_{ij} $(i, j \in \{1, \dots, n\})$. Entry a_{ij} is the exchange rate from w_i to w_j, i.e., for one unit of w_i one gets a_{ij} units of w_j.
Given a currency w_{i_0}, we want to find out whether there is a sequence $i_0, i_1, \dots, i_k$ such that we make profit if we exchange one unit of w_{i_0} to w_{i_1}, then to w_{i_2} etc. until w_{i_k} and then back to w_{i_0}.

(a) Translate this problem to a graph problem. That is, define a graph and a property which the graph fulfills if and only if there is a sequence of currencies as described above. *(4 Points)*

(b) Give an algorithm that decides in $\mathcal{O}(n^3)$ time steps whether there is a sequence of currencies as described above. Explain the correctness and runtime. *(6 Points)*

Hint: It is $a \cdot b > 1 \iff -\log a - \log b < 0$