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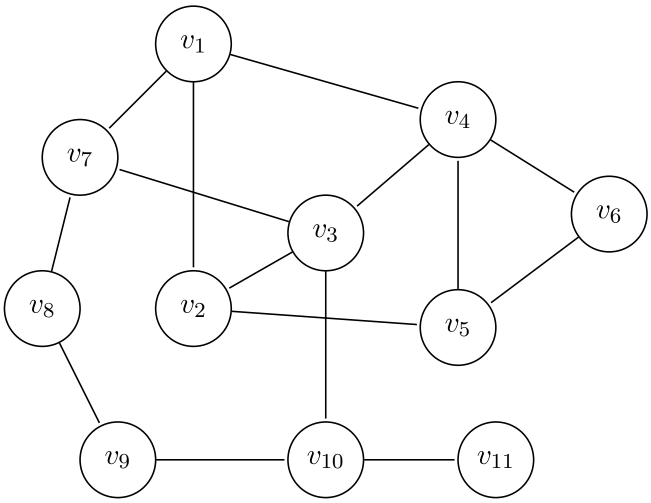
Algorithms and Datastructures
Summer Term 2024
Sample Solution Exercise Sheet 8

Due: Wednesday, January 8th, 2pm

Exercise 1: BFS *(5 Points)*

Given the following undirected graph G:



a) Provide G as an adjacency matrix. *(2 Points)*

b) Provide G as an adjacency list. *(2 Points)*

c) Perform a breadth-first search on G starting from node v_1. Write the order in which the nodes are marked (i.e., colored gray) in the algorithm. To obtain a deterministic result, always add the node with the smaller index to the FIFO-queue first, that is, v_i before v_j if $i < j$. *(3 Points)*

Sample Solution

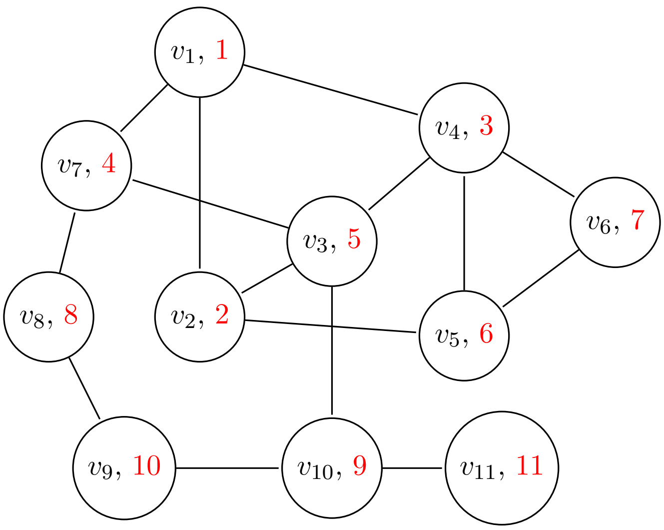
a)

$$
\begin{array}{c c}
\begin{array}{ccccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} \end{array} & \\
\begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
&
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \end{array}
\end{array}
$$

b)

- $v_1 : v_2, v_4, v_7$
- $v_2 : v_1, v_3, v_5$
- $v_3 : v_2, v_4, v_7, v_{10}$
- $v_4 : v_1, v_3, v_5, v_6$
- $v_5 : v_2, v_4, v_6$
- $v_6 : v_4, v_5$
- $v_7 : v_1, v_3, v_8$
- $v_8 : v_7, v_9$
- $v_9 : v_8, v_{10}$
- $v_{10} : v_3, v_9, v_{11}$
- $v_{11} : v_{10}$

c)



Exercise 2: DFS ***(6 Points)***

We define 2 timestamps for each node (as in Slide 29):

- $t_{v,1}$: Time when node v is colored gray by the DFS search

- $t_{v,2}$: Time when node v is colored black by the DFS search

Additionally, consider the following *directed* graph $G = (V, E)$ given with

- $V = \{u_1, u_2, u_3, u_4, u_5\}$
- $E = \{(u_1, u_2), (u_1, u_3), (u_2, u_3), (u_3, u_4), (u_4, u_1), (u_5, u_1), (u_5, u_3), (u_5, u_4)\}$

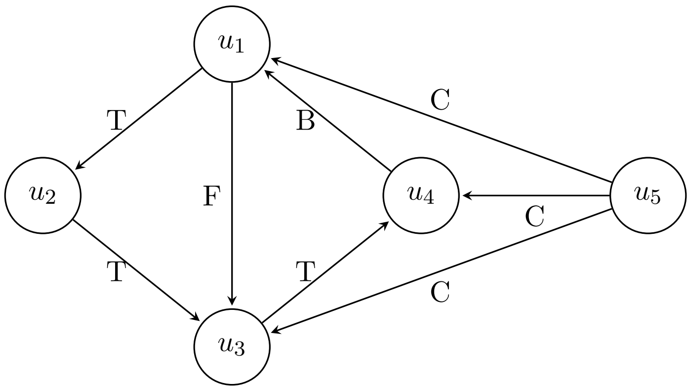
a) Draw G. *(2 Points)*

b) Write the processing interval $[t_{v,1}, t_{v,2}]$ for each node in G. Similar to part 1c), if multiple nodes could be visited next by the depth-first search, always choose the one with the smallest index (and thus we also start with u_1). *(2 Points)*

c) For each edge, indicate whether it is a **Tree Edge**, **Backward Edge**, **Forward Edge**, or **Cross Edge**. *(2 Points)*

Sample Solution

ac) We label a Tree Edge by T, a Backward Edge by B (Backward Edge), a Forward Edge by F and a Cross Edge by C:.



b)
- $u_1 : [1, 8]$
- $u_2 : [2, 7]$
- $u_3 : [3, 6]$
- $u_4 : [4, 5]$
- $u_5 : [9, 10]$

Exercise 3: Cycle search *(9 Points)*

a) How many edges m can an undirected connected graph with n nodes have at most? Justify your answer. *(2 Points)*

b) Show that every undirected connected graph which contains no cycle[1] has exactly $n - 1$ edges (where n is the number of nodes of the graph). *(4 Points)*
Hint: You can prove this statement, for example, by induction on $n \geq 1$.

c) Given an undirected connected graph $G = (V, E)$ with $n = |V|$. Provide an algorithm that decides in $\mathcal{O}(n)$ time whether G contains a cycle or not. Specify explicitly in which data structure G should be given. *(3 Points)*

[1] A cycle is a path $v_1, \ldots, v_k \in V$ in a graph where there is also an edge between the start and the end node, i.e., $\{v_1, v_k\} \in E$.

Sample Solution

a) A graph has the maximum number of edges when every node is connected to every other node. This means each node has a degree of $n-1$. We now fix an order of the nodes $v_1, ..., v_n$ and count the "not yet counted" edges for each. Thus, v_1 has exactly $n-1$ edges, v_2 still has $n-2$ edges (since the edge between v_1 and v_2 has already been counted), v_3 has $n-3$ edges, and so on. Therefore, we have:

$$m \leq \sum_{i=1}^{n}(n-i) = \sum_{i=1}^{n-1} i = \frac{(n-1)\cdot n}{2}$$

Another approach would be to calculate how many 2-element subsets there are of an n-element set. There are exactly $\binom{n}{2} = \frac{n!}{2!\cdot(n-2)!} = \frac{n\cdot(n-1)}{2!} = \frac{(n-1)\cdot n}{2}$.

b) A connected graph without cycles has exactly $n-1$ edges. Proof by induction.
Base case: For $n = 1$ the graph has no edges.
Induction hypothesis: Every such graph with $k \leq n-1$ nodes has $k-1$ edges.
Inductive step: We now show that the hypothesis also holds for a graph G with n nodes. Every graph G with n nodes can be composed of a node v which is connected to $l \geq 1$ disjoint subgraphs $G_1, ..., G_l$ of G. Since G is acyclic, each of these subgraphs is also acyclic, and the only connection between two subgraphs is through the node v. Without loss of generality, let us say that G_i has exactly n_i nodes (for each of these subgraphs). Since $n_i \leq n-1$ for all i, it follows from the induction hypothesis that G_i has exactly $n_i - 1$ edges. We can now calculate the number of edges m in G as follows:

$$m = deg(v) + \sum_{i=1}^{l}(n_i - 1) = l + \sum_{i=1}^{l} n_i - \sum_{i=1}^{l} 1 = \sum_{i=1}^{l} n_i = n-1$$

Here, $deg(v) = l$, since v is connected to each of the l subgraphs, and $\sum_{i=1}^{l} n_i = n-1$ because this is the sum over all nodes in G excluding v.

c) This task could theoretically be solved using either depth-first or breadth-first search. Here, we use breadth-first search and assume that G is given as an adjacency list. We perform the breadth-first search "normally", but we also record for each node v the node u from which it was first reached. This node u is called the **parent** of v. If v has a marked neighbor that is not its parent, then there is a cycle in the tree, and we return *false*. This procedure has the same runtime as breadth-first search, i.e., $O(n+m)$. If $m = O(n^2)$ is, as in task a), then the runtime is obviously too slow. However, we know from b) that if G is acyclic, it only has $n-1$ edges. We can therefore terminate the procedure after $n-1$ steps and return false if there are still unvisited nodes in the FIFO queue. Thus, the runtime is $O(n)$.
To justify why a cycle is found when node v has an already marked node, say w, as a neighbor: This would imply that there is a node s from which there is a path to both w and v. The edge between w and v connects these paths into a cycle.