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Algorithm Theory
Exercise Sheet 3

Due: Friday, 8th of November 2024, 10:00 am

Exercise 1: Covering Unit Intervals *(5 Points)*

We are given a set X of rational numbers. We want to find a minimum sized set S of unit intervals (i.e., intervals of the form $[a, a+1]$), such that each element in X is covered by at least one of these intervals from S. For example, let $X = \{1, 2.5, 3.1, 5\}$, then the set of unit intervals $S_1 := \{[0,1], [2,3], [3,4], [4.5, 5.5]\}$ covers all the elements in X, however, S_1 is not minimal. The minimum sized set contains only 3 intervals, for example $S_2 := \{[0,1], [2.3, 3.3], [4.5, 5.5]\}$.
Now, consider the following greedy algorithm $\mathcal{A}$:
In the first step $\mathcal{A}$ determines some $a \in \mathbb{Q}$ such that $[a, a+1]$ contains the maximum possible number of elements in X. This interval $[a, a+1]$ is added to S and the covered elements are deleted from X. $\mathcal{A}$ then recurse on the remaining elements and stops when X is empty.

a) Determine why $\mathcal{A}$ does not return an optimal solution. *(2 Points)*

b) Provide an efficient greedy algorithm to solve the problem. Argue why your algorithm is optimal. Use an exchange argument for your reasoning! *(3 Points)*

Exercise 2: Graph coloring *(7 Points)*

We say that a undirected graph $G = (V, E)$ has *degeneracy* k if every subgraph of G (and thus also G itself) has a vertex of degree at most k.

a) Show that given a graph G with *degeneracy* k can be colored with at most $k+1$ colors, i.e., there is a labeling of the nodes of G with 'colors' in $\{1, ..., k+1\}$ such that no neighbors are labeled with the same color. *(3 Points)*

b) There is the class of so-called planar graphs, for which it holds that $|E| \leq 3|V| - 6$. Show that every planar graph can be colored with at most 6 colors.[1] *(4 Points)*
Hint: Try to bound the degeneracy of planar graphs.

Exercise 3: Greedy TSP *(8 Points)*

Consider a symmetrical TSP instance. We have seen in the lecture that the *(greedy) nearest neighbor* approach can be arbitrarily bad. In this task we want to show that this is not true if we have some additional constrains. For that assume that all edges have weight either a or b with $0 < a < b$.

a) Prove that the nearest neighbor algorithm from the lecture is still not optimal. *(1 Point)*

[1] It is even known that every planar graph is 4-colorable.

b) Prove that the nearest neighbor algorithm from the lecture produces a TSP tour with a cost of at most a $\frac{a+b}{2a}$ factor from the optimal tour. In other words, show that if NN is the coast of the algorithm and OPT is the cost of the optimal tour that

$$\frac{NN}{OPT} \leq \frac{a+b}{2 \cdot a}.$$

Hint: Assume that OPT uses exactly k edges with weight a. Try to bound how often edges with weight a are used by NN in dependence of k. *(7 Points)*