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Algorithm Theory
Exercise Sheet 9

Due: Monday, 20th of November 2024, 10:00 am

Assumption: You may assume that calculations with real numbers can be performed with arbitrary precision in constant time.

Exercise 1: Unlock Achievement *(8 Points)*

There are N skills numbered 1 to N and M achievements numbered 1 to M.
Each skill has a positive integer level, and the initial level of every skill is 1.
You can pay a cost of C_i yen to increase the level of skill i by 1. You can do this as many times as you want.
Achievement i is achieved when the following condition is satisfied for every $j = 1, \ldots, N$, for which you will receive a reward of A_i yen.
Condition: The level of skill j is at least $L_{i,j}$.
Find the maximum possible total reward obtained minus the total cost required when appropriately choosing how to raise the skill levels in $O((NL + M)(NL + NM)^2)$ running time, where L is the maximum level a skill can have.

Hint: max(Reward-earned−Money-Spent) = − min(Money-Spent−Reward-earned) = All-rewards− min(Money-Spent + Reward-missed) . A cut defines a partition of the skills into bought/not bought and the skills into achieved/not achieved.

Exercise 2: Round Robin *(7 Points)*

There are N players numbered 1 through N participating in a round-robin tournament. Specifically:

- For every pair (i, j) where $1 \leq i < j \leq N$, Player i and Player j play a match against each other exactly once.
- This results in a total of
$$\binom{N}{2} = \frac{N(N-1)}{2}$$
matches.
- In every match, one player is declared the winner, and the other is the loser; there are no draws.

Match Results

- M matches have already been completed.
- In the i-th completed match, Player W_i won against Player L_i.

Objective

Determine all players who can still become the **unique winner** of the tournament after all remaining matches are played. A player is considered the unique winner if:

- The number of matches they win is strictly greater than the number of matches won by any other player.

Solve it in $O(N^5)$ running time.

Hint: Try to solve the question of "Is ith player the unique winner?". Try to create nodes for matches too not only for the players.

Exercise 3: Min of a minimum cut *(5 Points)*

You are given a flow network, i.e., a directed graph $D = (V, E)$ a source $s \in V$ and a sink $t \in V$, and two capacity functions on the edges $c_1, c_2 : E \to \mathbb{N}$. Your task is to find an $s-t$ cut in polynomial time that is minimum with respect to c_1 and among such cuts, it is minimum with respect to c_2. Prove that your algorithm is correct.

Hint: Try to create a new capacity function and prove that using the max flow on it will give the desired result.

Exercise 4: Problem for the exercise session *(0 Points)*

We have a grid with N horizontal rows and N vertical columns. Let (i, j) denote the square at the i-th row from the top and j-th column from the left. A character $c_{i,j}$ describes the color of the square (i, j). The character B means the square is painted black, W means the square is painted white, and ? means the square is not yet painted.
Takahashi will complete the black-and-white grid by painting each unpainted square black or white. Let the *zebraness* of the grid be the number of pairs of a black square and a white square sharing a side.
Find the maximum possible zebraness of the grid that Takahashi can achieve.