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Algorithm Theory
Exercise Sheet 13

Due: Friday, 31st of January, 2025, 10:00 am

Exercise 1: Generalized Contraction *(5 Points)*

Let $G = (V, E, w)$ be a weighted graph s.t. $w : V \to \mathbb{R}^+$. A cut (A, B) is a partition of V such that $V = A \cup B$, $A \cap B = \emptyset$, and $A, B \neq \emptyset$. We define the *weight of the cut* (A, B) to be the total edge weight crossing the cut.
correction after upoloading the sheet: the weight are on edges and not nodes i.e. we should define $w : E \to \mathbb{R}^+$.
Devise an algorithm that runs in $O(n^4 \log n)$ rounds and returns a minimum weighted cut *w.h.p.* Argue its correcntess and running time.

Exercise 2: Modified Contraction *(6 Points)*

Let's modify the contraction algorithm from the lecture in the following way: Instead of contracting a uniform random edge, we choose a uniform random pair of remaining nodes in each step and merge them. That is, as long as there are more than two nodes remaining, we choose two nodes $u \neq v$ uniformly at random and replace them by a new node w. For all edges $\{u, x\}$ and $\{v, x\}$ we add an edge $\{w, x\}$ and remove self-loops created at w.

(a) Give an example graph of size at least n where the above algorithm does not work well, that is, where the probability of finding a minimum cut is exponentially small in n (show that in the second part). *(2 Points)*

(b) Show that for your example the modified contraction algorithm has probability of finding a minimum cut at most a^n for some constant $a < 1$. *(4 Points)*

Exercise 3: Graph Connectivity *(9 Points)*

Let $G = (V, E)$ be a graph with n nodes and edge connectivity[1] $\lambda \geq \frac{16 \ln n}{\varepsilon^2}$ (where $0 < \varepsilon < 1$). Now every edge of G is removed with probability $\frac{1}{2}$. We want to show that the resulting graph $G' = (V, E')$ has connectivity $\lambda' \geq \frac{\lambda}{2}(1-\varepsilon)$ with probability at least $1 - \frac{1}{n}$. This exercise will guide you to this result.
Remark: If you don't succeed in a step you can use the result as a black box for the next step.

(a) Assume you have a cut of G with size $k \geq \lambda$. Show that the probability that the same cut in G' has size *strictly smaller* than $\frac{k}{2}(1-\varepsilon)$ is at most $e^{-\frac{\varepsilon^2 k}{4}}$. *(2 Points)*

(b) Let $k \geq \lambda$ be fixed. Show that the probability that at least one cut of G with size k becomes a cut of size *strictly smaller* than $\frac{k}{2}(1-\varepsilon)$ in G' is at most $e^{-\frac{\varepsilon^2 k}{8}}$.

Hint: You can use that for every $\alpha \geq 1$, the number of cuts of size at most $\alpha\lambda$ is at most $n^{2\alpha}$. (3 Points)

[1] The connectivity of a graph is the size of the smallest cut $(S, V \setminus S)$ in G.

(c) Show that for large n the probability that at least one cut of G with *any* size $k \geq \lambda$ becomes a cut of size *strictly smaller* than $\frac{k}{2}(1-\varepsilon)$ in G', is at most $\frac{1}{n}$.

Hint: Use another union bound. *(4 Points)*