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Algorithm Theory
Exercise Sheet 13

Due: Friday, 7th of February, 2025, 10:00 am

Exercise 1: Prof. Jot *(10 Points)*

Suppose you are given a list of N integers $L = [a_1, a_2, \ldots, a_N]$, a_i are positive numbers, and a positive integer C. The problem is to find a subset $S \subseteq \{1, 2, \ldots, N\}$ such that

$$T(S) = \sum_{i \in S} a_i \leq C,$$

and $T(S)$ is as large as possible.

(a)

Prof. Jot proposes the following greedy algorithm for obtaining an approximate solution to this maximization problem:

Algorithm 1 Greedy Algorithm for Bounded Set Sum

Require: List of integers $[a_1, \ldots, a_N]$, capacity C
Ensure: A subset S such that $T(S)$ is maximized under the constraint $T(S) \leq C$

```
S ← ∅, T ← 0
for i = 1 to N do
    if T + a_i ≤ C then
        S ← S ∪ {i}
        T ← T + a_i
return S
```

Show that Prof. Jot's algorithm is not a ρ-approximation algorithm for any fixed value ρ. (Use the convention that $\rho > 1$.)

(b)

Describe a 2-approximation algorithm for this maximization problem that runs in $O(N \log N)$ time.

Exercise 2: Miscellaneous Approximations *(10 Points)*

Let $G = (V, E)$ be an undirected connected graph. A set $D \subseteq V$ is called a *dominating set* if each node in V is either contained in D or adjacent to a node in D.
We consider the following randomized algorithm for d-regular graphs (i.e., graphs in which each node has exactly d neighbors).

Algorithm 2 `domset`(G)

```
D ← ∅
Each node joins D independently with probability p ← min{1, (c·ln n)/(d+1)} for some constant c ≥ 1
Each node that is neither in D nor has a neighbor in D joins D
return D
```

(a) The *minimum dominating set* problem asks to find a dominating set $D \subseteq V$ of minimum size. Show that for $c \geq 2$, the `domset` algorithm computes an $\mathcal{O}(\ln n)$-approximation of a minimum dominating set with probability at least $1 - \frac{2}{n}$. *(3 Points)*

(b) 1. An *independent set* is a set $I \subseteq V$ such that no two nodes in I share an edge in E. The *maximum independent set* problem asks to find an independent set of maximum size. Recall that the *minimum vertex cover* problem asks to find a vertex cover of minimum size. Now, show that both optimization problems are equivalent i.e. finding the minimum-size vertex cover is equivalent to finding the maximum-size independent set . *(2 Points)*

2. Show that the two problems are not equivalent in an approximation-preserving way, i.e it is not true that for all positive integer α, finding an α-approximate minimum vertex cover is equivalent to finding a α-approximate maximum independent set.

Hint: Give a counterexample by finding a family of graphs where one can easily obtain a 2-approximate minimum vertex cover, but this will equivalently find a very bad approximate maximum independent set. *(5 Points)*