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Algorithm Theory
Exercise Sheet 15

Due: Wednesday, 14th of February 2025, 11:59 pm

Exercise 1: Ticket Problem *(10 Bonus Points)*

A student from Freiburg is doing a one-year internship in Berlin, hence he will have to commute between the two cities. A train ticket from Freiburg to Berlin as well as from Berlin to Freiburg costs $p_0 > 0$ Euros. However, to save money there is a special ticket called *'RailCard50'* that is valid for the whole year and allows buying train tickets for half of the price. The *RailCard50* itself costs $p_1 = 10 \cdot p_0$ Euros. Consider this problem as an online problem, where the number of train rides $x \geq 1$ between these cities during the year is not known beforehand. So before each trip, if not bought yet, the student must make a decision on whether or not to buy the *RailCard50.*

(a) Describe the best offline strategy OPT (x is known beforehand) and give the costs as function depending on x. *(2 Points)*

(b) Assume the student decides on the online strategy ALG_1 (x is unknown), that is to buy the RailCard50 before the first train ride. Give an upper bound on the strict competitive ratio of ALG_1. *(3 Points)*

(c) Give an online strategy ALG_2 that is strictly $\frac{3}{2}$-competitive and prove it. *(5 Points)*

Exercise 2: Online Bin Packing *(10 Bonus Points)*

The *Online Bin Packing* problem is a variant of the *Knapsack* problem. Here we are given an unlimited number of bins, each with capacity 1. We get a sequence of items $x_1, x_2, ...$, in online fashion and are required to place them into the bins as we receive them (once placed we are not allowed to put an item into another bin). Each item x_i comes with an individual weight $0 < w_i \leq 1$. The goal is to minimize the number of used bins under the constraint that the sum of the weights of the items in one bin do not exceed its capacity.
In this task we consider the **First-Fit (FF)** online strategy: FF fixes the order of bins arbitrarily w.l.o.g. say $b_1, b_2, ...$, and places each item into the first bin (i.e., the bin with the smallest index) that has enough capacity left to hold the item.

(a) Show that FF is strictly 2-competitive. *(7 Points)*
Hint: Let C_i be the total weight of items in bin b_i. First show that for any given pair of bins b_i and b_j with $1 \leq i < j$ containing at least one element it is true that $C_i + C_j > 1$.

(b) Give a sequence of items for which the strictly competitive ratio of FF is no better than $\frac{3}{2}$. *(3 Points)*