Chapter 10
Parallel Algorithms

Algorithm Theory

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Sequential Algorithms

Classical Algorithm Design:

- One machine/CPU/process/… doing a computation

RAM (Random Access Machine):

- Basic standard model
- Unit cost basic operations
- Unit cost access to all memory cells

Sequential Algorithm / Program:

- Sequence of operations
  (executed one after the other)

Parallel and Distributed Algorithms

Today's computers/systems are not sequential:

- Even cell phones have several cores
- Future systems will be highly parallel on many levels
- This also requires appropriate algorithmic techniques

Goals, Scenarios, Challenges:

- Exploit parallelism to speed up computations
- Shared resources such as memory, bandwidth, …
- Increase reliability by adding redundancy
- Solve tasks in inherently decentralized environments
- …

Parallel and Distributed Systems

- Many different forms

- Processors/computers/machines/… communicate and share data through
  - Shared memory or message passing
- Computation and communication can be
  - Synchronous or asynchronous
- Many possible topologies for message passing
- Depending on system, various types of faults

Challenges

Algorithmic and theoretical challenges:

- How to parallelize computations
- Scheduling (which machine does what)
- Load balancing
- Fault tolerance
- Coordination / consistency
- Decentralized state
- Asynchrony
- Bounded bandwidth / properties of comm. channels
- …

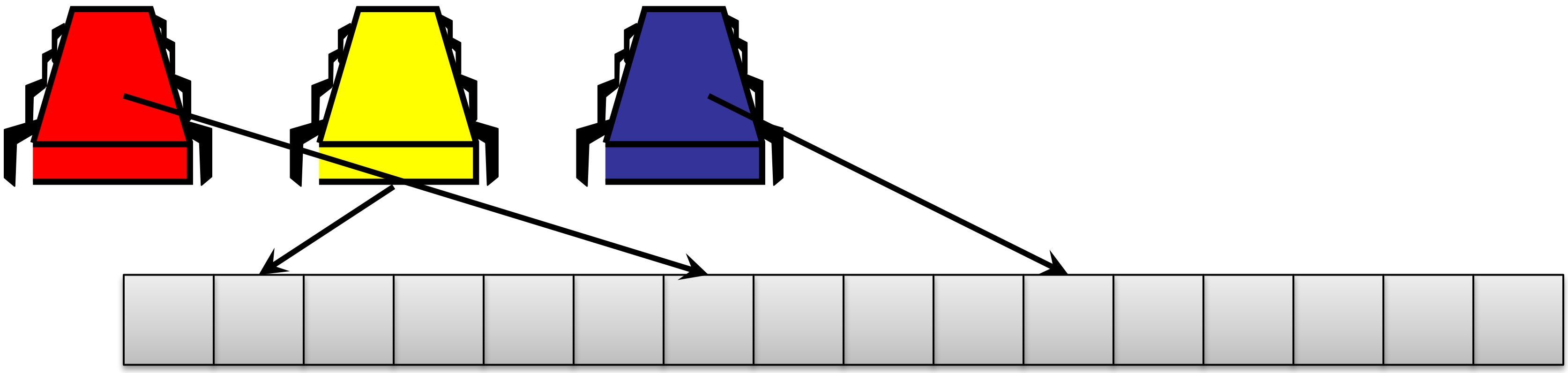
Models

- A large variety of models, e.g.:

- **PRAM** (Parallel Random Access Machine)
  - Classical model for parallel computations
- **Shared Memory**
  - Classical model to study coordination / agreement problems, distributed data structures, …
- **Message Passing** (fully connected topology)
  - Closely related to shared memory models
- Message Passing in **Networks**
  - Decentralized computations, large parallel machines, comes in various flavors…
- Computations in large clusters of powerful individual machines: Massively Parallel Computations (MPC)

PRAM

- Parallel version of RAM model
- p processors, shared random access memory



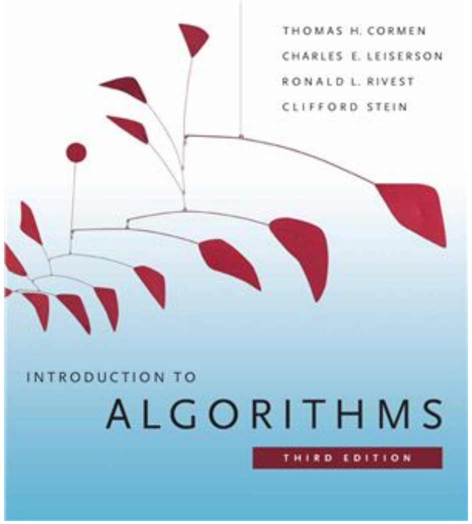
- Basic operations / access to shared memory cost 1
- Processor operations are synchronized
- Focus on parallelizing computation rather than cost of communication, locality, faults, asynchrony, …

Other Parallel Models

- **Message passing:** Fully connected network, local memory and information exchange using messages

- **Dynamic Multithreaded Algorithms:** Simple parallel programming paradigm
  - E.g., used in Cormen, Leiserson, Rivest, Stein (CLRS)

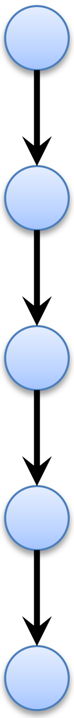
```
FIB(n)
if n < 2
    then return n
x ← spawn FIB(n − 1)
y ← spawn FIB(n − 2)
sync
return (x + y)
```



Parallel Computations

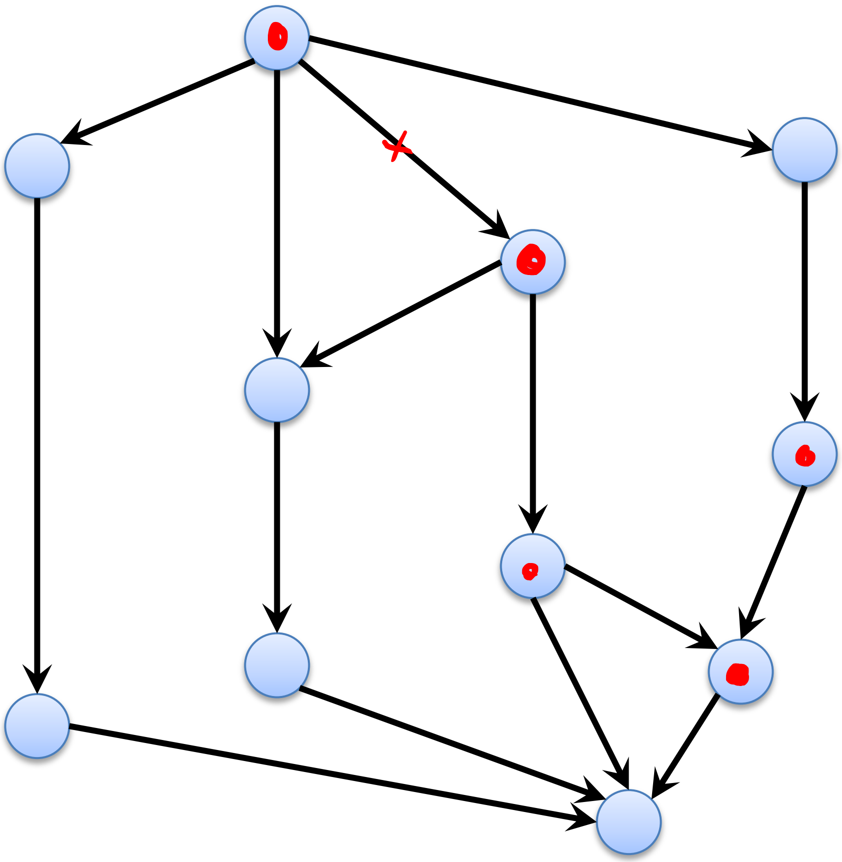
Sequential Computation:

- Sequence of operations



Parallel Computation:

- Directed Acyclic Graph (DAG)



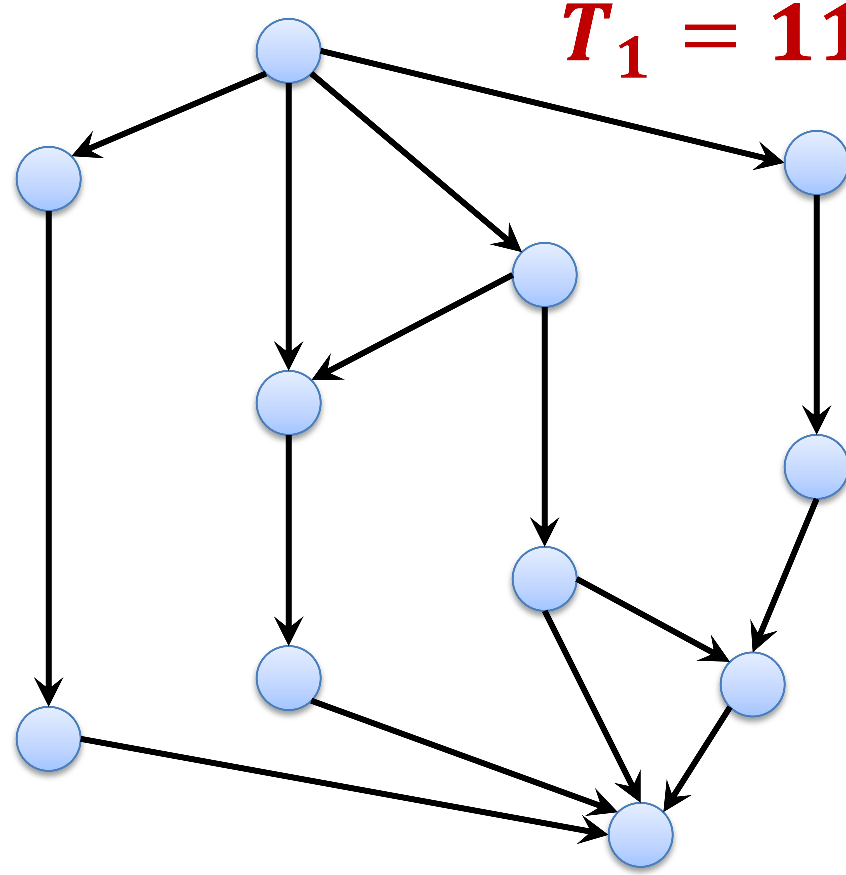
Parallel Computations

T_p: time to perform comp. with p procs

- T_1: **work** (total # operations)
  - Time when doing the computation sequentially
- T_∞: **critical path / span**
  - Time when parallelizing as much as possible
- **Lower Bounds**:

$$T_p \geq \frac{T_1}{p}, \qquad T_p \geq T_\infty$$

$T_1 = 11$



$T_\infty = 5$

Parallel Computations

$\boldsymbol{T_p}$: time to perform comp. with p procs

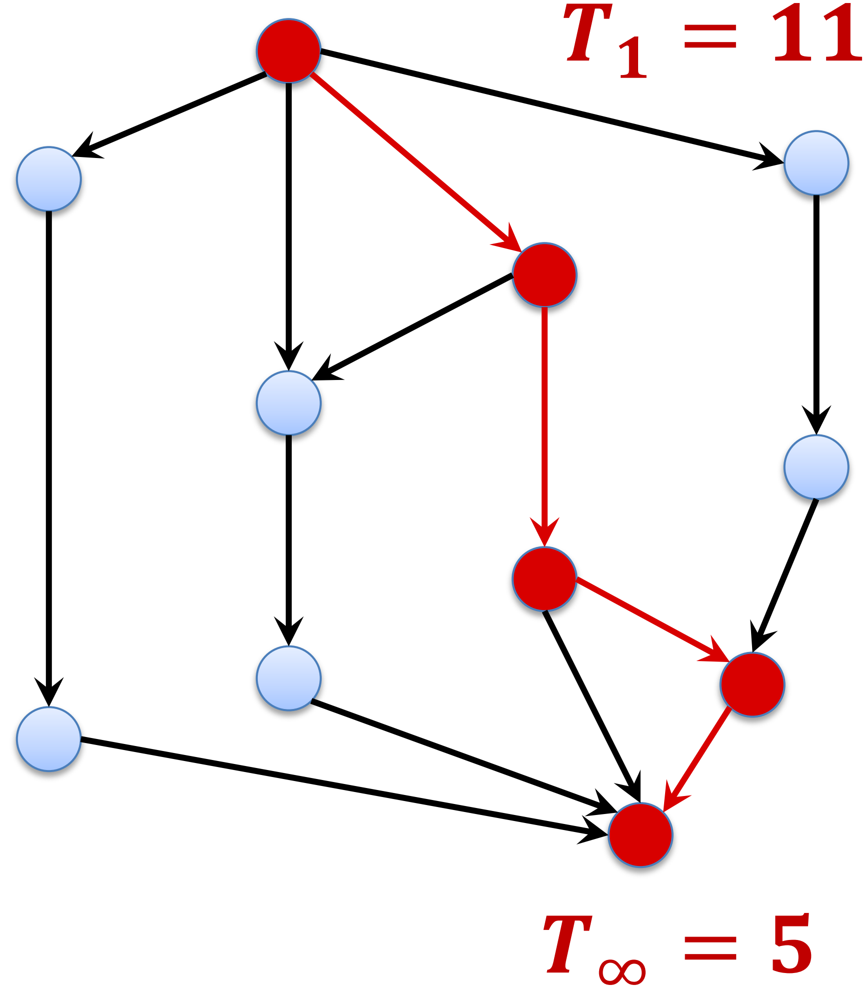
- **Lower Bounds**:

$$T_p \geq \frac{T_1}{p}, \qquad T_p \geq T_\infty$$

- **Parallelism**: $\frac{T_1}{T_\infty}$
  - maximum possible speed-up

- **Linear Speed-up**:

$$\frac{T_p}{T_1} = \Theta(p)$$



Scheduling

- How to assign operations to processors?

- Generally an online problem
  - When scheduling some jobs/operations, we do not know how the computation evolves over time

Greedy Scheduling:

- Order jobs/operations as they would be scheduled optimally with ∞ processors (topological sort of DAG)
  - Easy to determine: With ∞ processors, one always schedules all jobs/ops that can be scheduled
- Always schedule as many jobs/ops as possible
- Schedule jobs/ops in the same order as with ∞ processors
  - i.e., jobs that become available earlier have priority

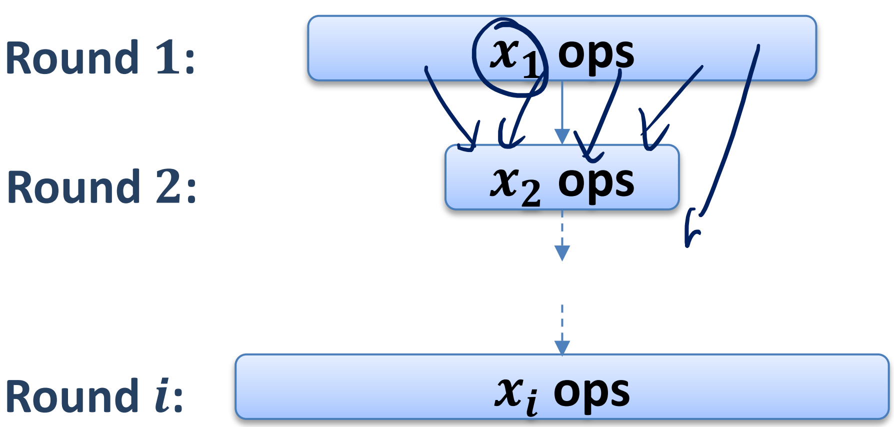
Brent's Theorem

Brent's Theorem: On p processors, a parallel computation can be performed in time

$$T_p \leq \frac{T_1 - T_\infty}{p} + T_\infty.$$

Proof:

- Greedy scheduling achieves this…
- #operations scheduled with ∞ processors in round i: x_i



Time with p processors:

$$\left\lceil \frac{x_1}{p} \right\rceil \leq \frac{x_1}{p} + \frac{p-1}{p} \leq \frac{x_1}{p} + 1$$

$$\left\lceil \frac{x_2}{p} \right\rceil \leq \frac{x_2}{p} + \frac{p-1}{p}$$

$$\left\lceil \frac{x_i}{p} \right\rceil \leq \frac{x_i}{p} + \frac{p-1}{p}$$

Brent's Theorem

Brent's Theorem: On p processors, a parallel computation can be performed in time

$$\boldsymbol{T_p \leq \frac{T_1 - T_\infty}{p} + T_\infty}.$$

Proof:

- Greedy scheduling achieves this…
- #operations scheduled with ∞ processors in round i: $\boldsymbol{x_i}$
- Time t_i to schedule the x_i ops of round i with p processors:

$$t_i = \left\lceil \frac{x_i}{p} \right\rceil \leq \frac{x_i}{p} + \frac{p-1}{p}$$

- Overall time with p processors:

$$T_p^{(G)} \leq \sum_{i=1}^{T_\infty} t_i \leq \sum_{i=1}^{T_\infty} \left(\frac{x_i}{p} + \frac{p-1}{p} \right) = \frac{1}{p} \cdot \sum_{i=1}^{T_\infty} x_i + T_\infty \cdot \frac{p-1}{p} = \frac{T_1 - T_\infty}{p} + T_\infty$$

Brent's Theorem

Brent's Theorem: On p processors, a parallel computation can be performed in time

$$T_p \leq \frac{T_1 - T_\infty}{p} + T_\infty.$$

Corollary: Greedy is a 2-approximation algorithm for scheduling.

Opt. complexity with p processors : T_p^*

$$\left.\begin{array}{l} T_p^* \geq \frac{T_1}{p} \\ T_p^* \geq T_\infty \end{array}\right\} \quad T_p^{(G)} \leq \frac{T_1}{p} + T_\infty \leq 2 \cdot T_p^*$$

Corollary: As long as the number of processors $p = O(T_1/T_\infty)$, it is possible to achieve a linear speed-up.

PRAM

Back to the PRAM:

- Shared random access memory, synchronous computation steps
- The PRAM model comes in variants…

EREW (exclusive read, exclusive write):

- Concurrent memory access by multiple processors is not allowed
- If two or more processors try to read from or write to the same memory cell concurrently, the behavior is not specified

CREW (concurrent read, exclusive write):

- Reading the same memory cell concurrently is OK
- Two concurrent writes to the same cell lead to unspecified behavior
- This is the first variant that was considered (already in the 70s)

PRAM

The PRAM model comes in variants…

CRCW (concurrent read, concurrent write):

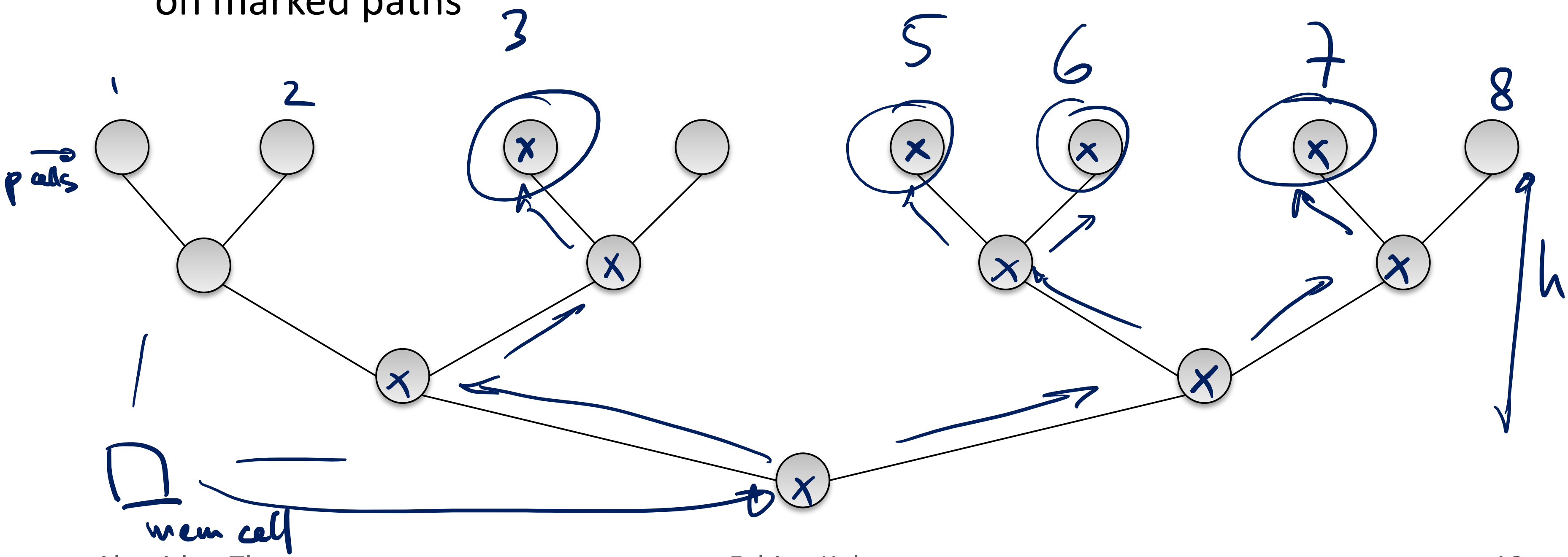
- Concurrent reads and writes are both OK
- Behavior of concurrent writes has to specified
  - Weak CRCW: concurrent write only OK if all processors write 0
  - Common-mode CRCW: all processors need to write the same value
  - Arbitrary-winner CRCW: adversary picks one of the values
  - Priority CRCW: value of processor with highest ID is written
  - Strong CRCW: largest (or smallest) value is written
- The given models are ordered in strength:

  weak $\leq$ **common-mode** $\leq$ **arbitrary-winner** $\leq$ **priority** $\leq$ **strong**

Some Relations Between PRAM Models

Theorem: A parallel computation that can be performed in time t, using p proc. on a strong CRCW machine, can also be performed in time $O(t \log p)$ using p processors on an EREW machine.

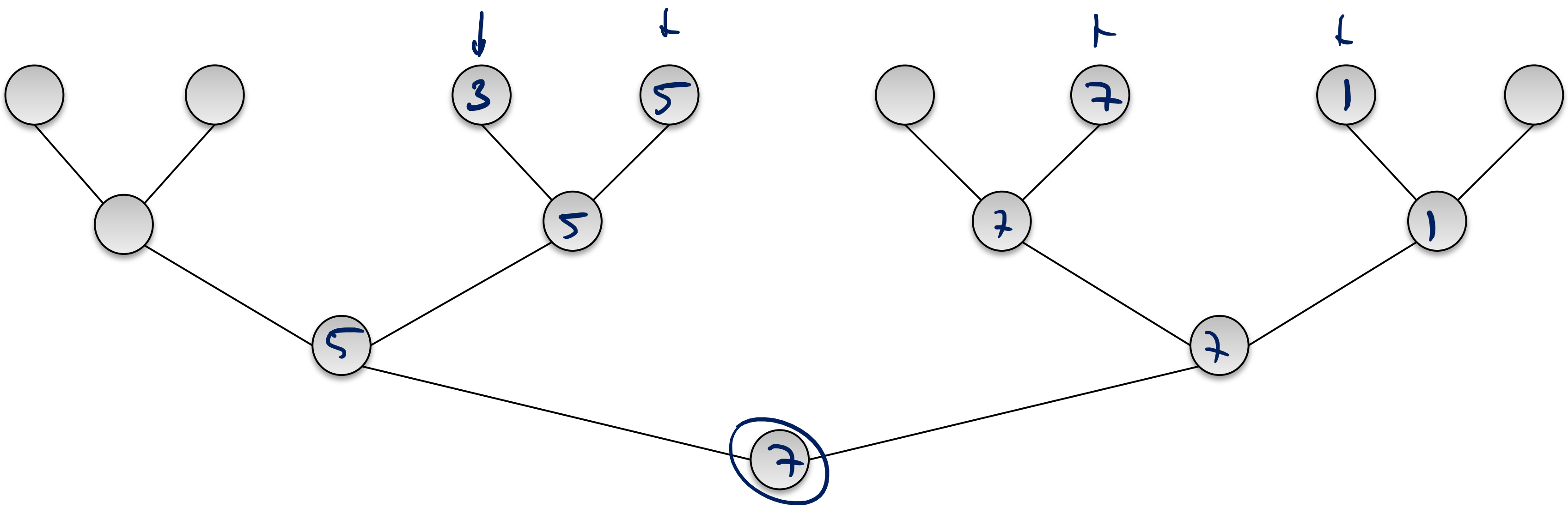
- Each (parallel) CRCW step can be simulated by $O(\log p)$ EREW
- For each register, add $O(p)$ additional registered, logically connected to a binary tree
- **Reading the register:** mark from leaves to root, then copy value from register on marked paths



Some Relations Between PRAM Models

Theorem: A parallel computation that can be performed in time t, using p proc. on a strong CRCW machine, can also be performed in time $O(t \log p)$ using p processors on an EREW machine.

- Each (parallel) CRCW step can be simulated by $O(\log p)$ EREW
- For each register, add $O(p)$ additional registered, logically connected to a binary tree
- **Writing the register:** start at leaves and propagate the winning value towards the root



Some Relations Between PRAM Models

Theorem: A parallel computation that can be performed in time t, using p proc. on a strong CRCW machine, can also be performed in time $O(t \log p)$ using p processors on an EREW machine.

- Each (parallel) step on the CRCW machine can be simulated by $O(\log p)$ steps on an EREW machine

Theorem: A parallel computation that can be performed in time t, using p probabilistic processors on a strong CRCW machine, can also be performed in expected time $O(t \log p)$ using $O(p/\log p)$ processors on an arbitrary-winner CRCW machine.

- The same simulation turns out more efficient in this case

Some Relations Between PRAM Models

Theorem: A computation that can be performed in time t, using p processors on a strong CRCW machine, can also be performed in time $O(t)$ using $O(p^2)$ processors on a weak CRCW machine

Proof:

- Strong: largest value wins, weak: only concurrently writing 0 is OK
- Processes:
  - Both machines use processes $1, \dots, p$
  - Weak machine: additional procs q_{ij} for every pair (i, j), $1 \leq i < j \leq p$
- Additional memory cells of weak CRCW machine:

  $\forall i \in \{1, \dots, p\}$: flag f_i, value v_i, address a_i (all initialized to 0)

- If process i wants to write value x to memory cell c:

  set $\boldsymbol{f_i := 1,\ v_i := x,\ a_i := c}$

Some Relations Between PRAM Models

Theorem: A computation that can be performed in time t, using p processors on a strong CRCW machine, can also be performed in time $O(t)$ using $O(p^2)$ processors on a weak CRCW machine

Proof:

- Strong: largest value wins, weak: only concurrently writing 0 is OK
- If process i wants to write value x to memory cell c:

  set $\boldsymbol{f_i := 1,\ v_i := x,\ a_i := c}$

$\forall i, j$: q_{ij} reads cells $f_i, f_j, v_i, v_j, a_i, a_j$ *(concurrent reads are OK)*

if $f_i = f_j = 1\ \wedge a_i = a_j$ then *(i and j write to same addr.)*

if $v_i \geq v_j$ then $f_j := 0$ *(set flag to 0 if value loses)*

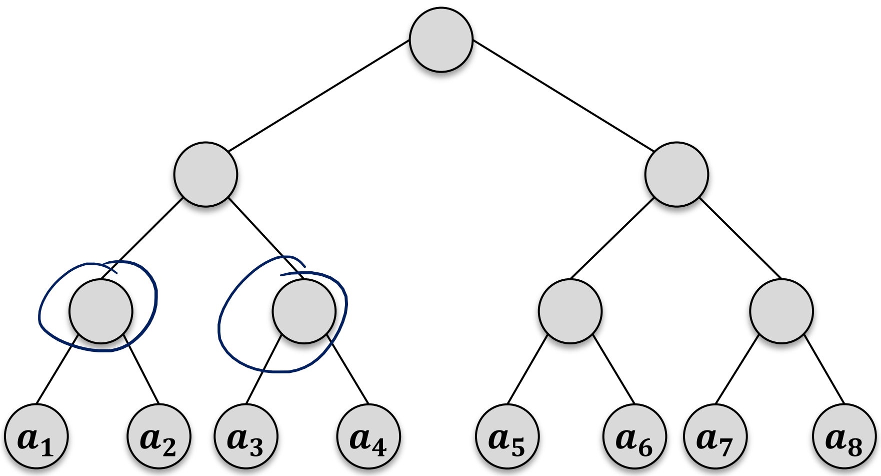
else $f_i := 0$ *(concurrent writes of 0 OK)*

Computing the Maximum

Given: n values

Goal: find the maximum value

Observation: The maximum can be computed in parallel by using a binary tree (even on an EREW PRAM).



Work $T_1 = O(n)$

Depth $T_\infty = O(\log n)$

Time $T_p = O\left(\frac{n}{p} + \log n\right)$

Linear speed-up ($T_p = O(T_1 \,/\, p)$) as long as $p = O(n/\log n)$

Computing the Maximum

Observation: On a strong CRCW machine, the maximum of a n values can be computed in $O(1)$ time using n processors

- Each value is concurrently written to the same memory cell

Lemma: On a weak CRCW machine, the maximum of n integers between 1 and $\sqrt{n}$ can be computed in time $O(1)$ using $O(n)$ proc.

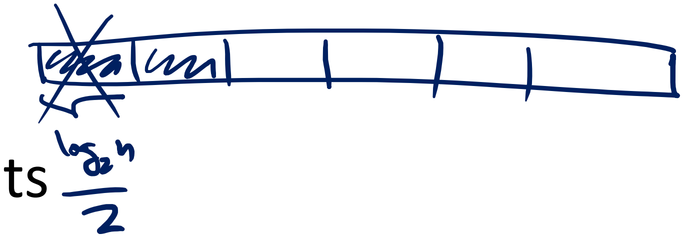
Proof:

- We have $\sqrt{n}$ memory cells $f_1, \dots, f_{\sqrt{n}}$ for the possible values
- Initialize all $f_i := 1$
- For the n values $x_1, \dots, x_n$, processor j sets $f_{x_j} := 0$
  - Since only zeroes are written, concurrent writes are OK
- Now, $f_i = 0$ iff value i occurs at least once
- Strong CRCW machine: max. value in time $O(1)$ w. $O(\sqrt{n})$ proc.
- Weak CRCW machine: time $O(1)$ using $O(n)$ proc. (prev. lemma)

Computing the Maximum

Theorem: If each value can be represented using $O(\log n)$ bits, the maximum of n (integer) values can be computed in time $O(1)$ using $O(n)$ processors on a weak CRCW machine.

Proof:



- First look at $\frac{\log_2 n}{2}$ highest order bits
- The maximum value also has the maximum among those bits
- There are only $\sqrt{n}$ possibilities for these bits
- max. of $\frac{\log_2 n}{2}$ highest order bits can be computed in $O(1)$ time
- For those with largest $\frac{\log_2 n}{2}$ highest order bits, continue with next block of $\frac{\log_2 n}{2}$ bits, …

Prefix Sums

- The following works for any associative binary operator $\oplus$:

associativity: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

All-Prefix-Sums: Given a sequence of n values $a_1, \dots, a_n$, the all-prefix-sums operation w.r.t. $\oplus$ returns the sequence of prefix sums:

$$s_1, s_2, \dots, s_n = a_1, a_1 \oplus a_2, a_1 \oplus a_2 \oplus a_3, \dots, a_1 \oplus \cdots \oplus a_n$$

- Can be computed efficiently in parallel and turns out to be an important building block for designing parallel algorithms

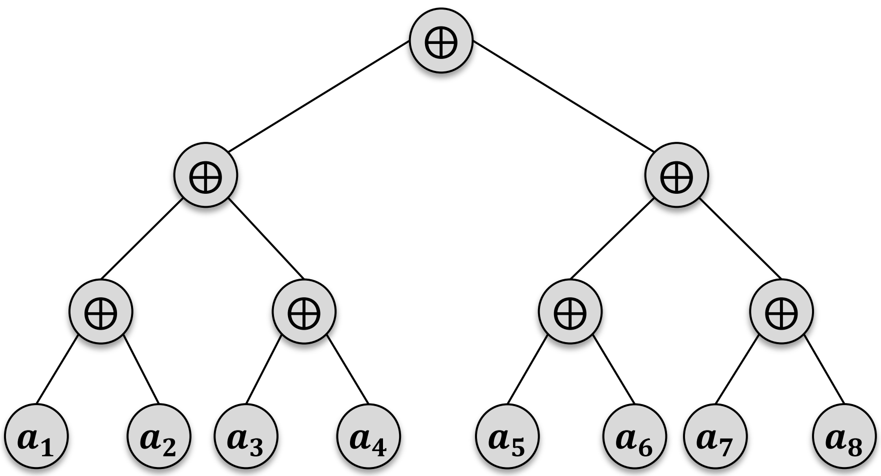
Example: Operator: $+$, input: $a_1, \dots, a_8 = 3, 1, 7, 0, 4, 1, 6, 3$

$s_1, \dots, s_8 =$ 3, 4, 11, 11, 15, 16, 22, 25

Computing the Sum

- Let's first look at $s_n = a_1 \oplus a_2 \oplus \cdots \oplus a_n$

- Parallelize using a binary tree:



Work $T_1 = O(n)$

Depth $T_\infty = O(\log n)$

Time (p procs) $T_p = O\left(\frac{n}{p} + \log n\right)$

Linear speed-up ($T_p = O(T_1 / p)$) as long as $p = O(n/\log n)$

Computing the Sum

Lemma: The sum $s_n = a_1 \oplus a_2 \oplus \cdots \oplus a_n$ can be computed in time $O(\log n)$ on an EREW PRAM. The total number of operations (total work) is $O(n)$.

Proof:

- Use a binary tree of height $O(\log n)$
- Tree has $O(n)$ nodes (each computes one sum of two values)

Corollary: The sum s_n can be computed in time $O(\log n)$ using $O(n/\log n)$ processors on an EREW PRAM.

Proof:

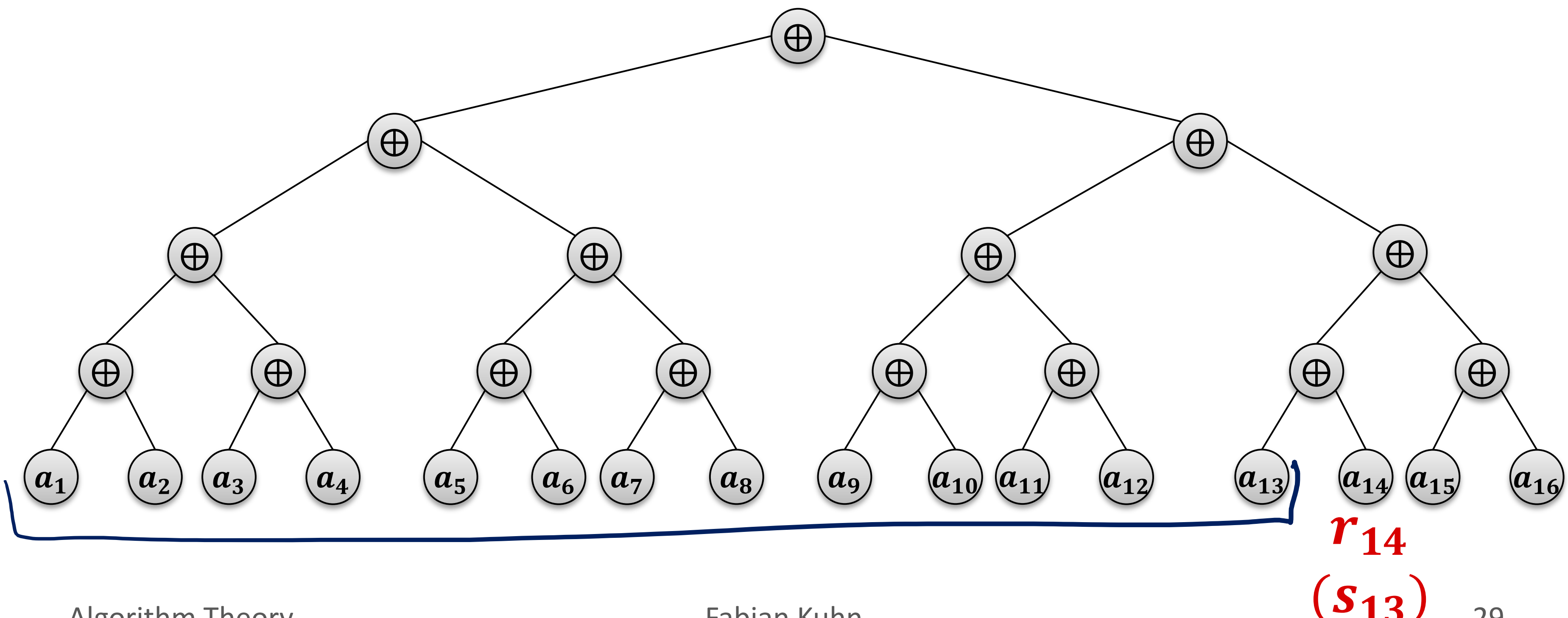
- Follows from Brent's theorem ($T_1 = O(n)$, $T_\infty = O(\log n)$)

Getting The Prefix Sums

- Instead of computing the sequence $s_1, s_2, \dots, s_n$ let's compute $r_1, \dots, r_n = 0, s_1, s_2, \dots, s_{n-1}$ (0: neutral element w.r.t. $\oplus$)

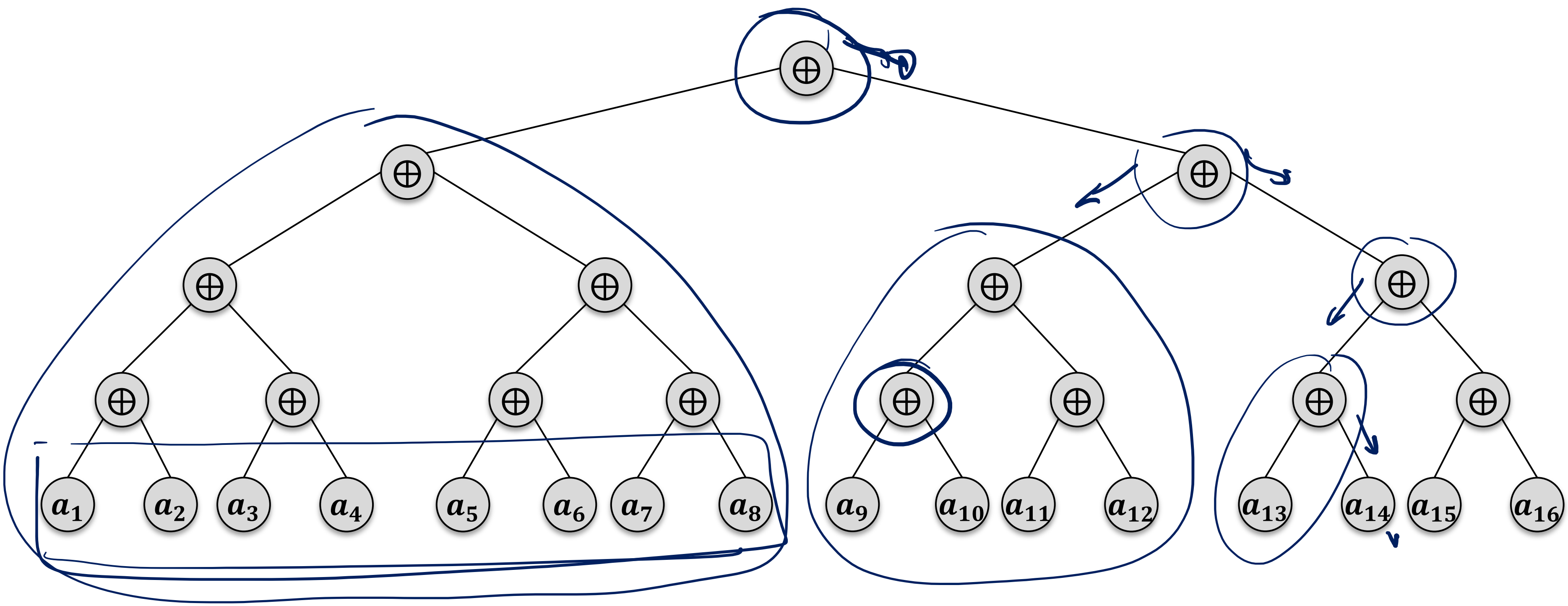
$$r_1, \dots, r_n = 0, a_1, a_1 \oplus a_2, \dots, a_1 \oplus \cdots \oplus a_{n-1}$$

- Together with s_n, this gives all prefix sums
- Prefix sum $r_i = s_{i-1} = a_1 \oplus \cdots \oplus a_{i-1}$:



Getting The Prefix Sums

Claim: The prefix sum $r_i = a_1 \oplus \cdots \oplus a_{i-1}$ is the sum of all the leaves in the left sub-tree of ancestor u of the leaf v containing a_i such that v is in the right sub-tree of u.



Computing The Prefix Sums

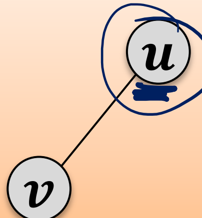
For each node v of the binary tree, **define $r(v)$ as follows:**

- $r(v)$ is the sum of the values a_i at the leaves in all the left sub-trees of ancestors u of v such that v is in the right sub-tree of u.

For a leaf node v holding value a_i: $\boldsymbol{r(v) = r_i = s_{i-1}}$

For the root node: $\boldsymbol{r(\text{root}) = 0}$

For all other nodes v:

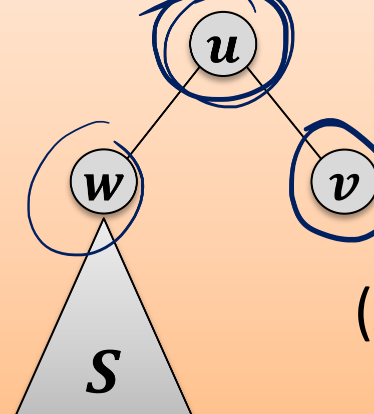


v is the left child of u:

$$r(v) = r(u)$$

v is the right child of u:

(u has left child w)

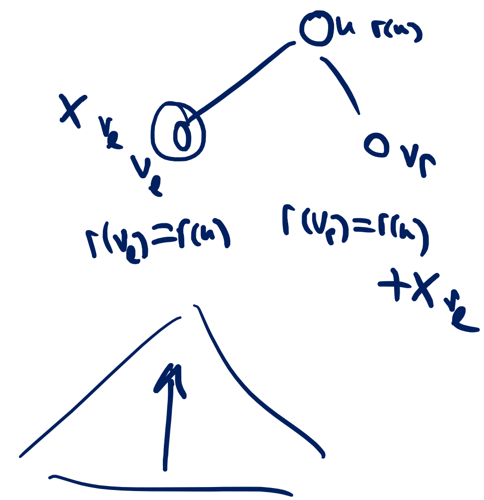


$$r(v) = r(u) + S$$

(S: sum of values in sub-tree of w)

Computing The Prefix Sums

- leaf node v holding value a_i: $\boldsymbol{r(v) = r_i = s_{i-1}}$
- root node: $\boldsymbol{r(\mathbf{root}) = 0}$
- Node v is the left child of u: $r(v) = r(u)$
- Node v is the right child of u: $r(v) = r(u) + S$
  - Where: $S =$ sum of values in left sub-tree of u

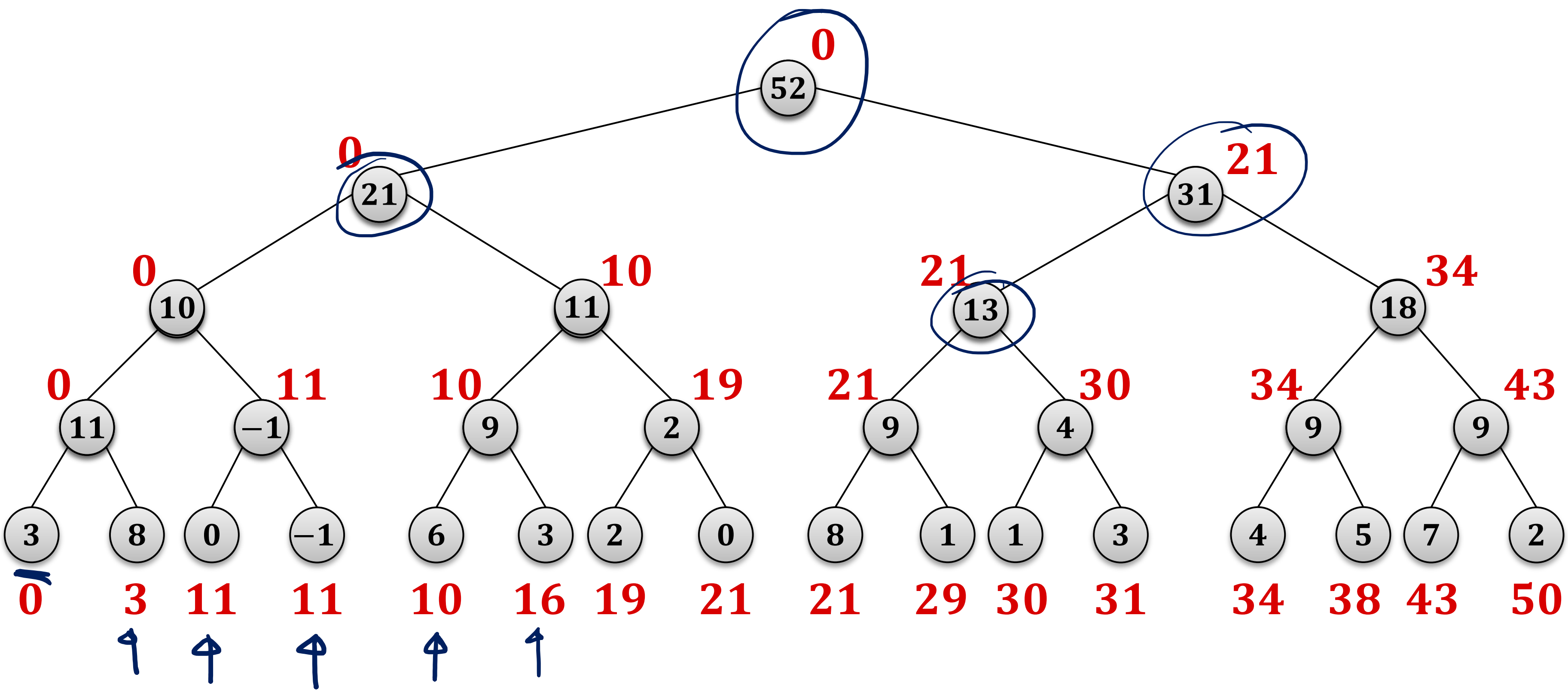


Algorithm to compute values $r(v)$:

1. Compute sum of values in each sub-tree (bottom-up)
   - Can be done in parallel time $O(\log n)$ with $O(n)$ total work
2. Compute values $r(v)$ top-down from root to leaves:
   - To compute the value $r(v)$, only $r(u)$ of the parent u and the sum of the left sibling (if v is a right child) are needed
   - Can be done in parallel time $O(\log n)$ with $O(n)$ total work

Example

1. Compute sums of all sub-trees
   – Bottom-up (level-wise in parallel, starting at the leaves)
2. Compute values $r(v)$
   – Top-down (starting at the root)



Computing Prefix Sums

Theorem: Given a sequence $a_1, \dots, a_n$ of n values, all prefix sums $s_i = a_1 \oplus \cdots \oplus a_i$ (for $1 \leq i \leq n$) can be computed in time $O(\log n)$ using $O(n/\log n)$ processors on an EREW PRAM.

Proof:

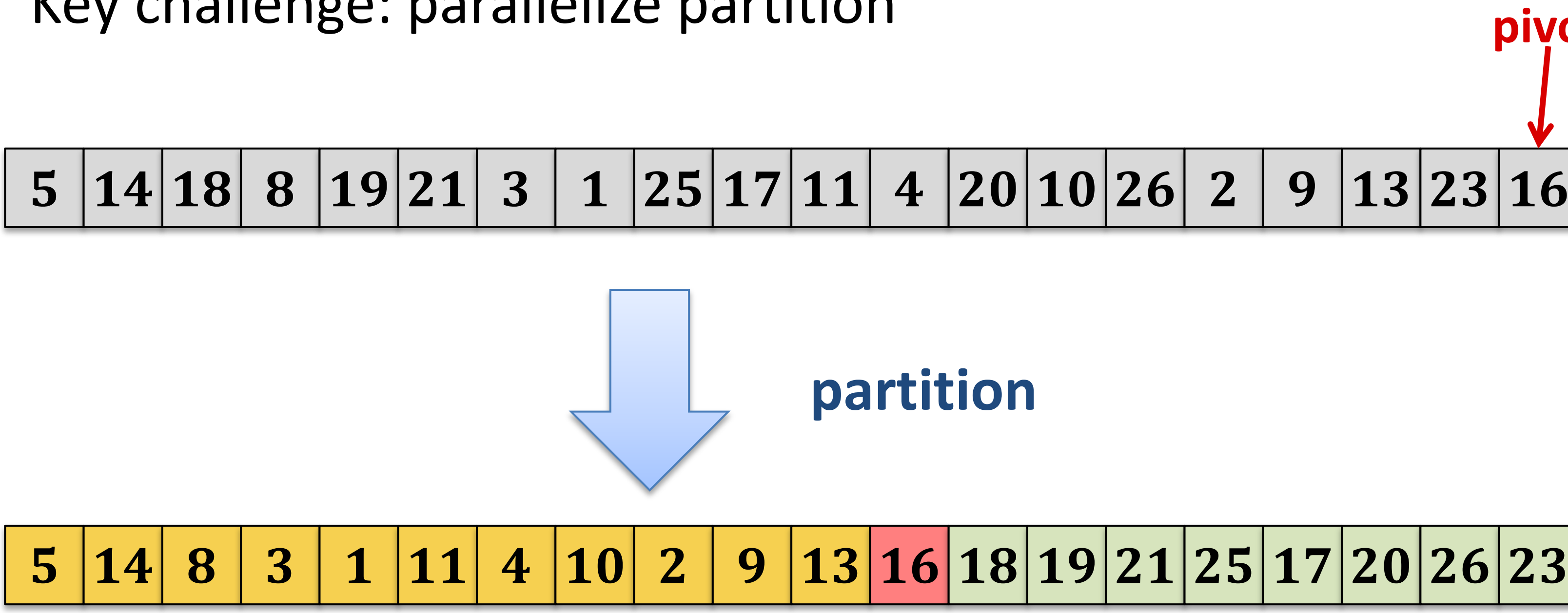
- Computing the sums of all sub-trees can be done in parallel in time $O(\log n)$ using $O(n)$ total operations.
- The same is true for the top-down step to compute the $r(v)$
- The theorem then follows from Brent's theorem:

$$T_1 = O(n), \qquad T_\infty = O(\log n) \qquad \Longrightarrow \qquad T_p < T_\infty + \frac{T_1}{p}$$

Remark: This can be adapted to other parallel models and to different ways of storing the value (e.g., array or list)

Parallel Quicksort

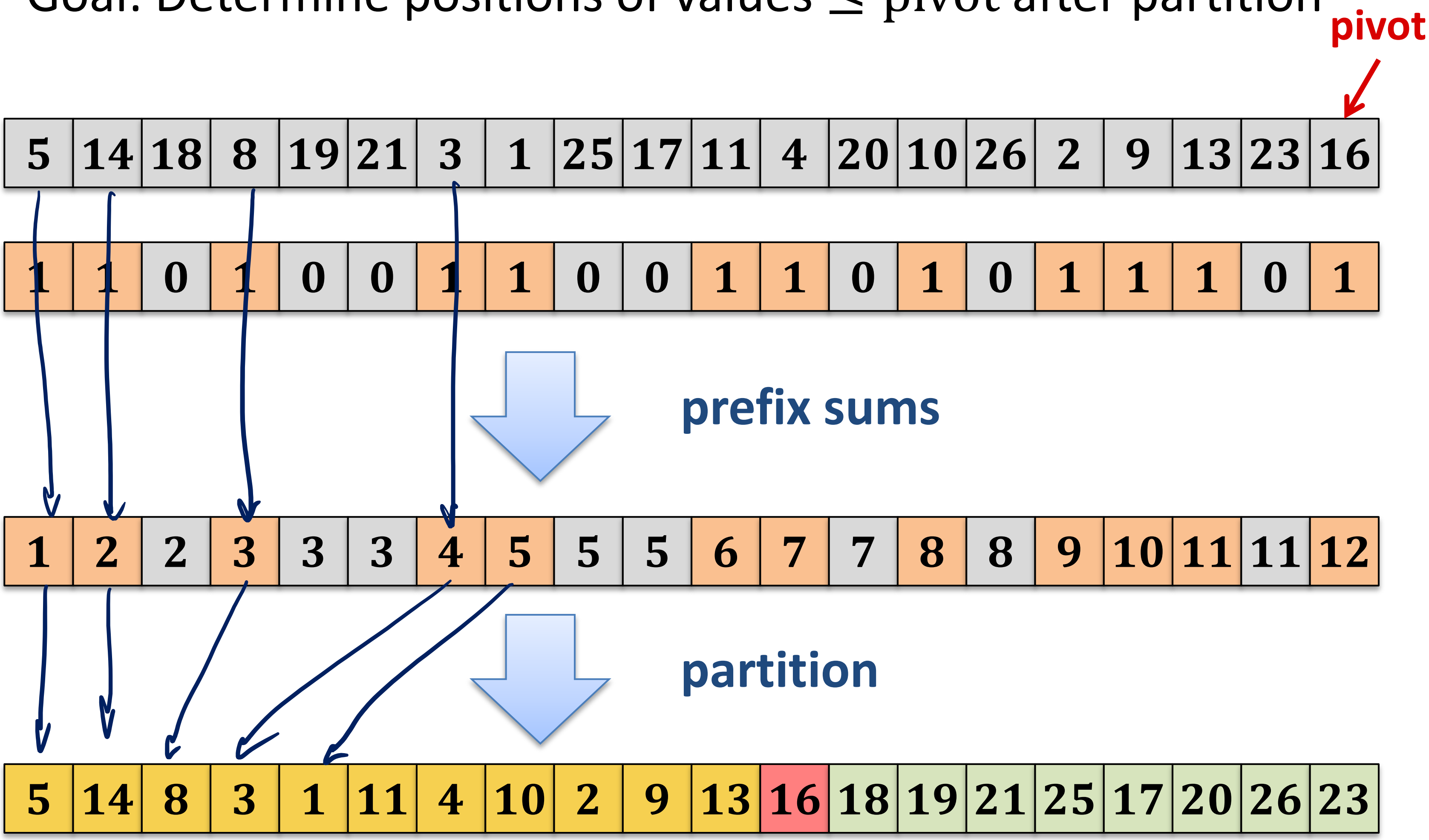
- Key challenge: parallelize partition



- How can we do this in parallel?
- For now, let's just care about the values $\leq$ pivot
- What are their new positions

Using Prefix Sums

- Goal: Determine positions of values $\leq$ pivot after partition



Partition Using Prefix Sums

- The positions of the entries $>$ pivot can be determined in the same way
- **Prefix sums:** $T_1 = O(n)$, $\quad T_\infty = O(\log n)$

  ↙ O(log n) on EREW
- **Remaining computations:** $T_1 = O(n)$, $\quad T_\infty = O(1)$
- **Overall:** $T_1 = O(n)$, $\quad T_\infty = O(\log n)$

Lemma: The partitioning of quicksort can be carried out in parallel in time $O(\log n)$ using $O\left(\frac{n}{\log n}\right)$ processors.

Proof:

- By Brent's theorem: $T_p \leq \frac{T_1}{p} + T_\infty$

Applying to Quicksort

Theorem: On an EREW PRAM, using p processors, randomized quicksort can be executed in time T_p (in expectation and with high probability), where

$$T_p = O\left(\frac{n \log n}{p} + \log^2 n\right).$$

Proof:

- Work $\quad T_1 = O(n \log n)$
- Depth/Span $\quad T_\infty = O(\log^2 n)$

Remark:

- We get optimal (linear) speed-up w.r.t. to the sequential algorithm for all $p = O(n/\log n)$.

Other Applications of Prefix Sums

- Prefix sums are a very powerful primitive to design parallel algorithms.
  - Particularly also by using other operators than "+"

Example Applications:

- Lexical comparison of strings
- Add multi-precision numbers
- Evaluate polynomials
- Solve recurrences
- Radix sort / quick sort
- Search for regular expressions
- Implement some tree operations
- …