Algorithm Theory

Chapter 7
Randomized Algorithms

Part I:
Contention Resolution

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Randomization

Randomized Algorithm:

- An algorithm that uses (or can use) random coin flips in order to make decisions

We will see: randomization can be a powerful tool to

- Make algorithms faster
- Make algorithms simpler
- Make the analysis simpler
  - Sometimes it's also the opposite…
- Allow to solve problems (efficiently) that cannot be solved (efficiently) without randomization
  - True in some computational models (e.g., for distributed algorithms)
  - Not clear in the standard sequential model

Contention Resolution

A simple starter example (from distributed computing)

- Allows to introduce important concepts
- … and to repeat some basic probability theory

Setting:

- n processes, 1 resource
  (e.g., communication channel, shared database, …)
- There are time slots $1,2,3, \dots$
- In each time slot, only one process can access the resource
- All processes need to regularly access the resource
- If process i tries to access the resource in slot t:
  - Successful iff no other process tries to access the resource in slot t

Algorithm

Algorithm Ideas:

- Accessing the resource deterministically seems hard
  - need to make sure that processes access the resource at different times
  - or at least: often only a single process tries to access the resource

- **Randomized solution:**
  In each time slot, each process tries with probability p.

Analysis:

- How large should p be?
- How long does it take until some process x succeeds?
- How long does it take until all processes succeed?
- What are the probabilistic guarantees?

Analysis

Events:

- $\mathcal{A}_{x,t}$: process x tries to access the resource in time slot t
  - Complementary event: $\overline{\mathcal{A}_{x,t}}$

$$\mathbb{P}(\mathcal{A}_{x,t}) = p, \qquad \mathbb{P}\left(\overline{\mathcal{A}_{x,t}}\right) = 1 - p$$

- $\mathcal{S}_{x,t}$: process x is successful in time slot t

$$\mathcal{S}_{x,t} = \mathcal{A}_{x,t} \cap \left(\bigcap_{y \neq x} \overline{\mathcal{A}_{y,t}}\right)$$

x is successful if
- x tries to access resource **and**
- no other process tries to access resource

- **Success probability** (for process x):

$$\mathbb{P}(\mathcal{S}_{x,t}) = \mathbb{P}(\mathcal{A}_{x,t}) \cdot \prod_{y \neq x} \mathbb{P}\left(\overline{\mathcal{A}_{y,t}}\right) = p \cdot (1-p)^{n-1}$$

Choose p that maximizes $\mathbb{P}(\mathcal{S}_{x,t})$

Fixing p

- $\mathbb{P}(\mathcal{S}_{x,t}) = p(1-p)^{n-1}$ is maximized for

$$\boldsymbol{p = \frac{1}{n}} \quad \Longrightarrow \quad \mathbb{P}(\mathcal{S}_{x,t}) = \underbrace{\frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}}_{\text{converges to } 1/e \text{ for } n \to \infty}.$$

- **Asymptotics:**

$$\text{For } n \geq 2: \quad \frac{1}{4} \leq \left(1-\frac{1}{n}\right)^{n} < \frac{1}{e} < \left(1-\frac{1}{n}\right)^{n-1} \leq \frac{1}{2}$$

- **Success probability:**

$$\boldsymbol{\frac{1}{en} < \mathbb{P}(\mathcal{S}_{x,t}) \leq \frac{1}{2n}}$$

Time Until First Success

$q \coloneqq \mathbb{P}(\mathcal{S}_{x,t}) = p(1-p)^{n-1}$

Random Variable T_x:

- $T_x = t$ if proc. x is successful in slot t for the first time
- **Distribution:**

$$\mathbb{P}(T_x = 1) = q,\ \mathbb{P}(T_i = 2) = (1-q)\cdot q, \dots$$

$$\mathbb{P}(T_x = t) = (1-q)^{t-1}\cdot q$$

- T_x is geometrically distributed with parameter

$$q = \mathbb{P}(\mathcal{S}_{x,t}) = \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1} > \frac{1}{en}.$$

- **Expected time** until first success:

$$\mathbb{E}[\boldsymbol{T_x}] \coloneqq \sum_{t=1}^{\infty} t\cdot\mathbb{P}(T_x = t) = \frac{\mathbf{1}}{\boldsymbol{q}} < \boldsymbol{en}$$

Time Until First Success

Failure Event $\mathcal{F}_{x,t}$: Process x does not succeed in time slots $1, \dots, t$

$$\mathcal{F}_{x,t} = \bigcap_{r=1}^{t} \overline{\mathcal{S}_{x,r}}$$

- The events $\mathcal{S}_{x,t}$ are independent for different t:

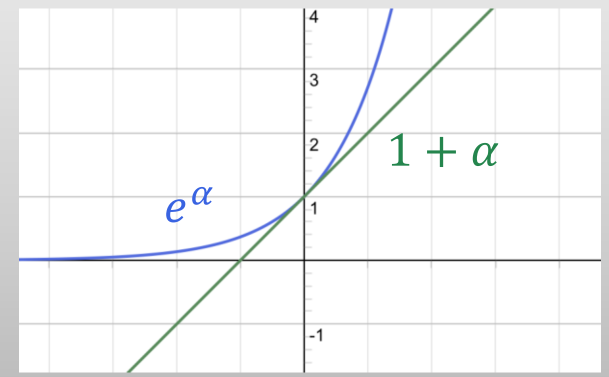
$$\mathbb{P}(\mathcal{F}_{x,t}) = \mathbb{P}\left(\bigcap_{r=1}^{t} \overline{\mathcal{S}_{x,r}}\right) = \prod_{r=1}^{t} \mathbb{P}\left(\overline{\mathcal{S}_{x,r}}\right) = \left(1 - \mathbb{P}(\mathcal{S}_{x,1})\right)^t = (1-q)^t$$

- We know that $\mathbb{P}(\mathcal{S}_{x,r}) > {}^1/_{en}$:

$$\mathbb{P}(\mathcal{F}_{x,t}) < \left(1 - \frac{1}{en}\right)^t \leq e^{-t/en}$$

$$1 - {}^1/_{en} \leq e^{-1/en}$$

$\forall \boldsymbol{\alpha} \in \mathbb{R} : (\mathbf{1} + \boldsymbol{\alpha}) \leq \boldsymbol{e}^{\boldsymbol{\alpha}}$



Time Until First Success

No success by time t: $\mathbb{P}(\mathcal{F}_{x,t}) < e^{-t/en}$

$t = \lceil en \rceil$: $\mathbb{P}(\mathcal{F}_{x,t}) < 1/e$

- Generally if $t = \Theta(n)$: constant success probability

$e^{c \cdot \ln n} = (e^{\ln n})^c = n^c$

$t = \lceil c \cdot en \cdot \ln n \rceil$: $\mathbb{P}(\mathcal{F}_{x,t}) < 1/e^{c \cdot \ln n} = 1/n^c$

- For success probability $1 - 1/n^c$, we need $t = \Theta(n \log n)$.
- We say that x succeeds **with high probability** in $O(n \log n)$ time.

With probability $\geq 1 - \frac{1}{n^c}$ for any constant $c > 0$.

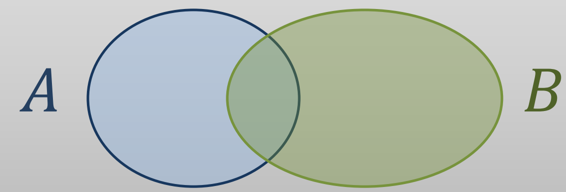
Choice of c only affects the hidden constant in the big-O notation.

Time Until All Processes Succeed

Event $\mathcal{F}_t$: some process has not succeeded by time t

$$\mathcal{F}_t = \bigcup_{x=1}^{n} \mathcal{F}_{x,t}$$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \leq \mathbb{P}(A) + \mathbb{P}(B)$$



Union Bound: For events $\mathcal{E}_1, \dots, \mathcal{E}_k$,

$$\mathbb{P}\left(\bigcup_{x}^{k} \mathcal{E}_x\right) \leq \sum_{x}^{k} \mathbb{P}(\mathcal{E}_x)$$

Probability that not all processes have succeeded by time t:

$$\mathbb{P}(\mathcal{F}_t) = \mathbb{P}\left(\bigcup_{x=1}^{n} \mathcal{F}_{x,t}\right) \leq \sum_{x=1}^{n} \mathbb{P}(\mathcal{F}_{x,t}) < n \cdot e^{-t/en}.$$

Time Until All Processes Succeed

Claim: With high probability, all processes succeed in the first $O(n \log n)$ time slots.

Proof:

- $\mathbb{P}(\mathcal{F}_t) < n \cdot e^{-t/en}$
- Set $t = \lceil (c+1) \cdot en \cdot \ln n \rceil$

$$\mathbb{P}(\mathcal{F}_t) < n \cdot e^{-\frac{(c+1) \cdot en \cdot \ln n}{en}} = n \cdot e^{-(c+1) \cdot \ln n} = n \cdot \frac{1}{n^{c+1}} = \frac{1}{n^c}$$

Remarks:

- $\Theta(n \log n)$ time slots are necessary for all processes to succeed even with reasonable (constant) probability
- $\Theta(n \log n)$ time slots are also necessary in expectation for all processes to succeed at least once.