universität freiburg

Algorithm Theory – WS 2024/25

Chapter 2 : Greedy Algorithms

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Greedy Algorithms

- No clear definition, but essentially:

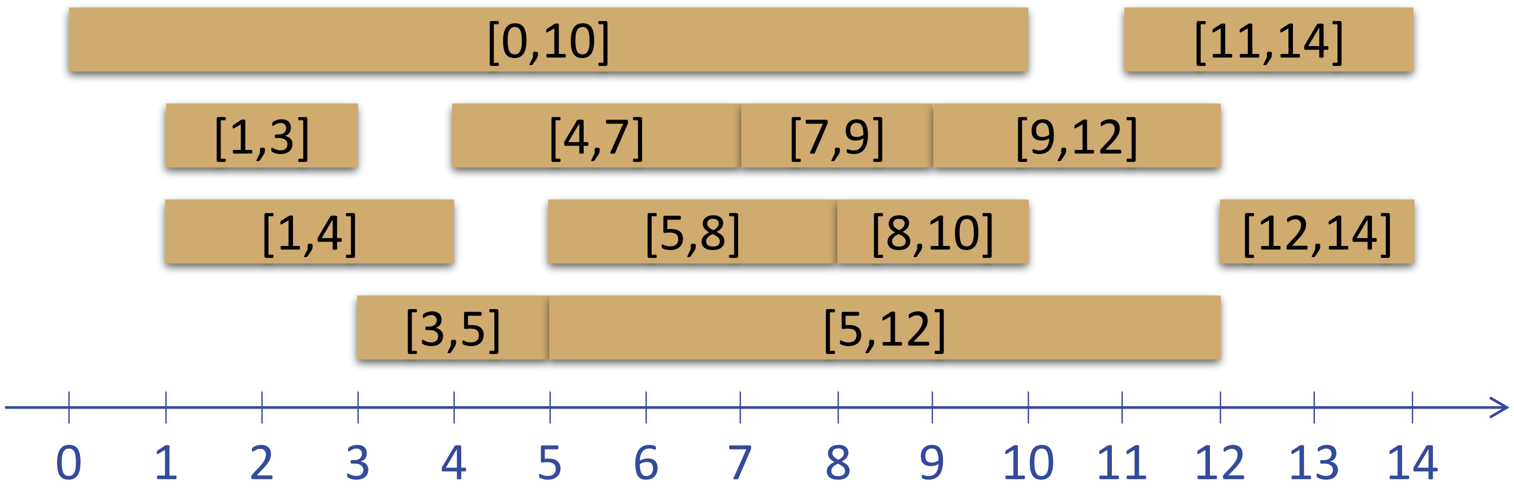
In each step make the choice that looks best at the moment!

- Depending on problem, greedy algorithms can give
  - Optimal solutions
  - Close to optimal solutions
  - No (reasonable) solutions at all
- If it works, very interesting approach!
  - And we might even learn something about the structure of the problem

Goal: Improve understanding where it works (mostly by examples)

Interval Scheduling

- **Given:** Set of intervals, e.g. [0,10],[1,3],[1,4],[3,5],[4,7],[5,8],[5,12],[7,9],[9,12],[8,10],[11,14],[12,14]



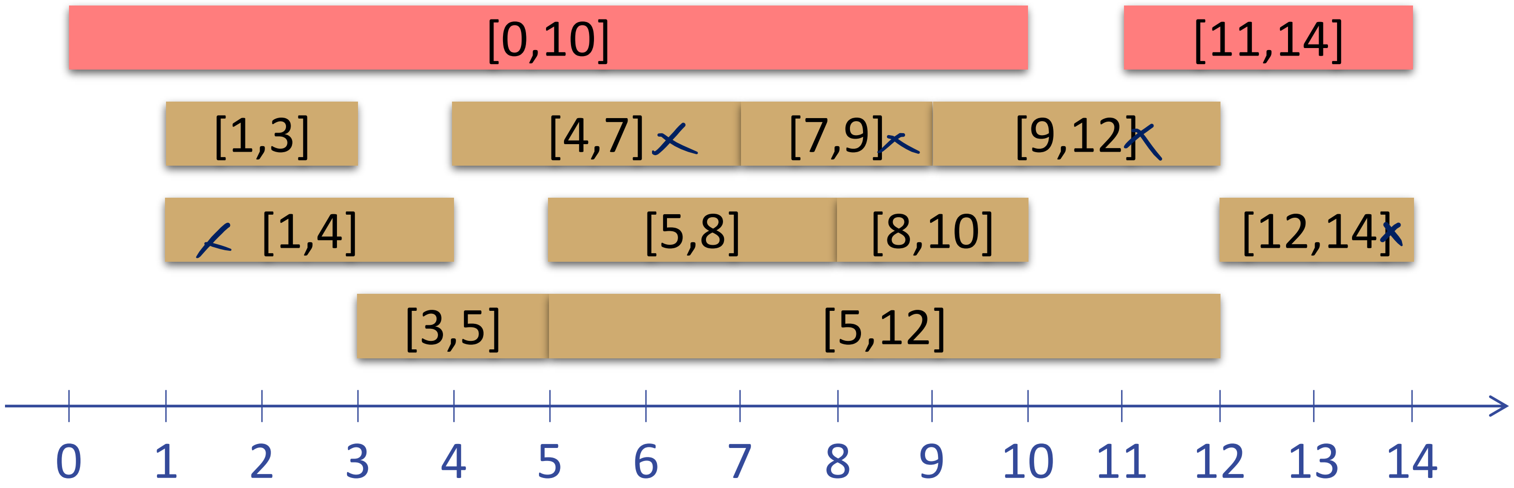
- **Goal:** Select largest possible non-overlapping set of intervals
  - For simplicity: overlap at boundary ok
    (i.e., [4,7] and [7,9] are non-overlapping)

- **Example:** Intervals are room requests; satisfy as many as possible

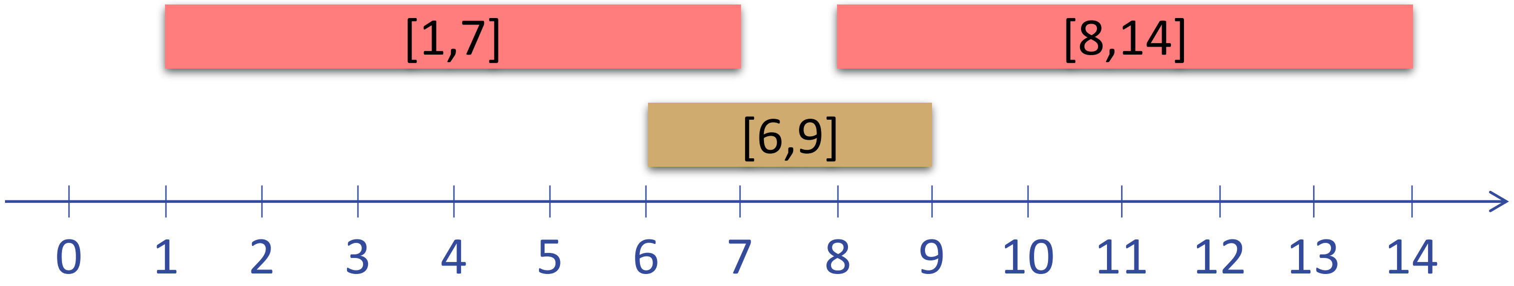
Greedy Algorithms

- Several possibilities…

Choose first available interval:

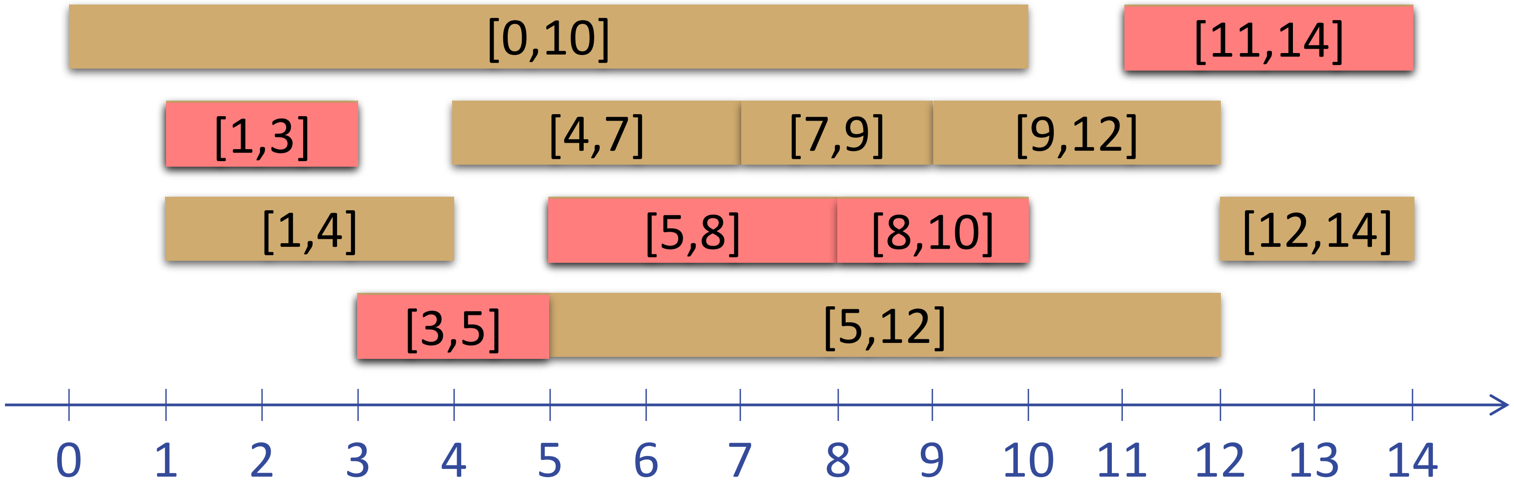


Choose shortest available interval:



Greedy Algorithms

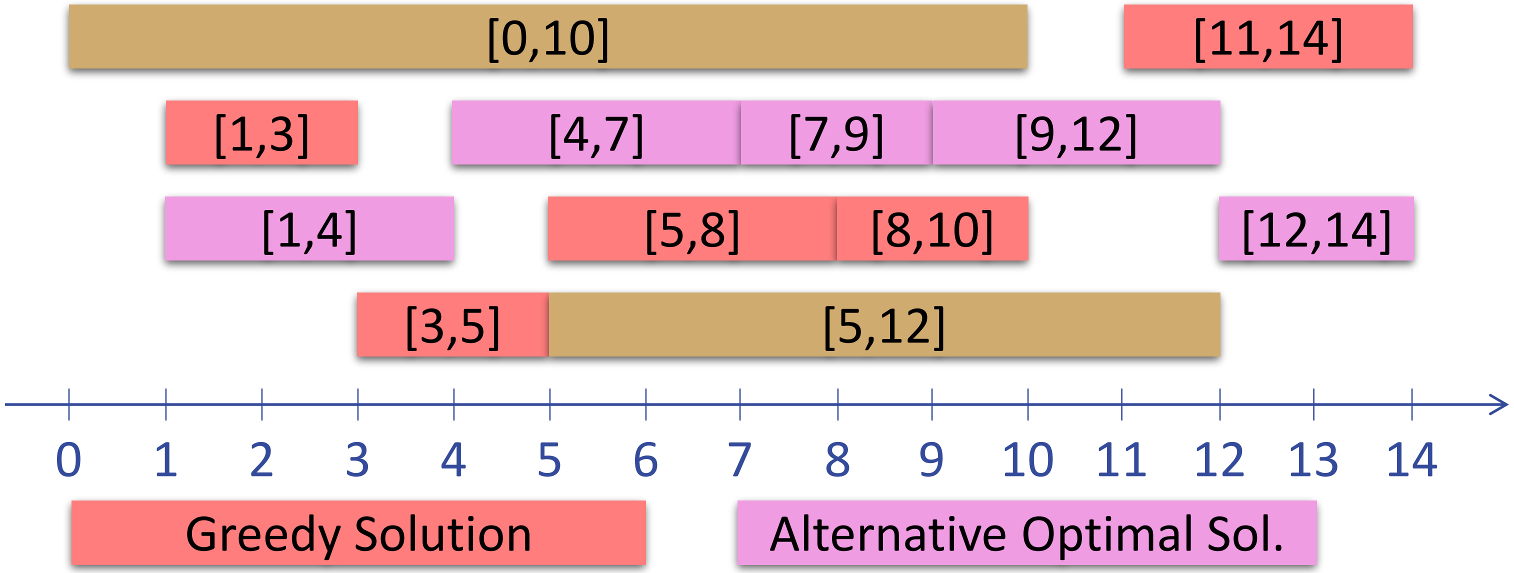
Choose available request with earliest finishing time:



R ≔ set of all requests; S ≔ empty set;
while R is not empty **do**
    choose $r \in R$ with smallest finishing time
    add r to S
    delete all requests from R that are not compatible with r
end   // S is the solution

Earliest Finishing Time is Optimal

- Let O be the set of intervals of an optimal solution
- Can we show that $S = O$?
  - No…



- Show that $|S| = |O|$.

Or alternatively: $|S| \geq |O|$ for any other solution O.

Greedy Stays Ahead

- Greedy solution S:

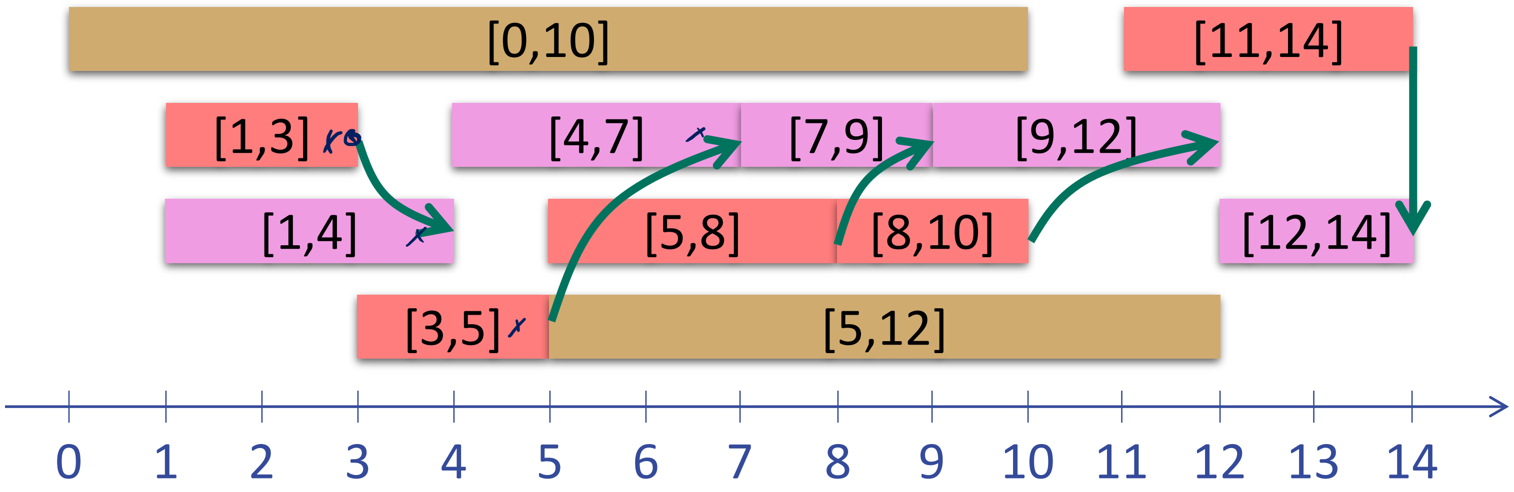
$$[a_1, b_1], [a_2, b_2], \dots, [a_{|S|}, b_{|S|}], \qquad \text{where } b_i \leq a_{i+1}$$

- Some optimal solution O:

$$[a_1^*, b_1^*], [a_2^*, b_2^*], \dots, [a_{|O|}^*, b_{|O|}^*], \qquad \text{where } b_i^* \leq a_{i+1}^*$$

- Define $b_i := \infty$ for $i > |S|$ and $b_i^* := \infty$ for $i > |O|$

Claim: $\forall i \geq 1,\ b_i \leq b_i^*$ $\qquad \Longrightarrow |S| \geq |O|$ because $b_{|O|} \leq b_{|O|}^* < \infty$



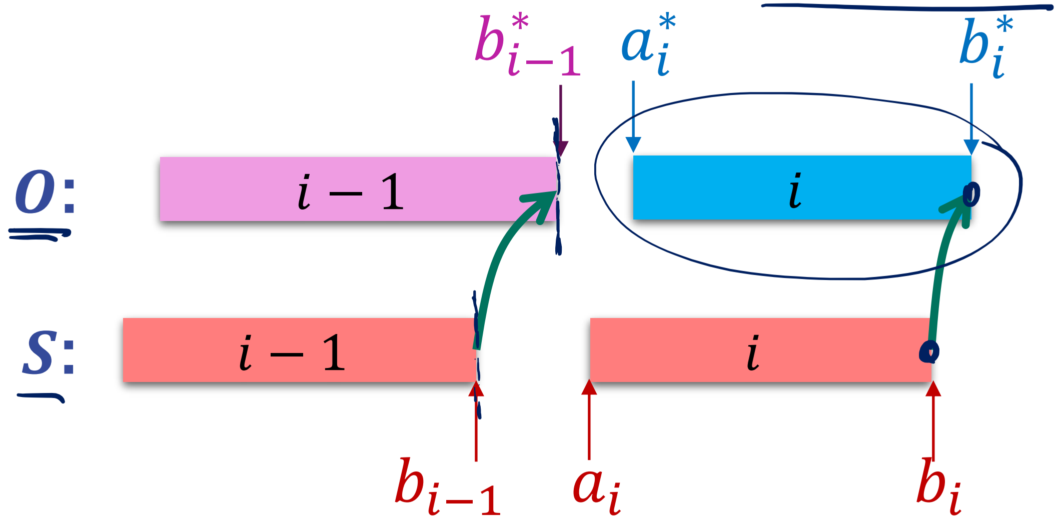
Greedy Stays Ahead

Claim: For all $i \geq 1$, $b_i \leq b_i^*$

Proof (by induction on i):

Base case $i = 1$: $\boldsymbol{b_1 \leq b_1^*}$ ✓

Step $i - 1 \rightarrow i$: Induction Hypothesis: $\boldsymbol{b_{i-1} \leq b_{i-1}^*}$



Need to show that $b_i \leq b_i^*$:

Blue interval is available to greedy algorithm because

$$b_{i-1} \leq b_{i-1}^* \leq a_i^*$$

Greedy would prefer blue interval if $b_i^* < b_i$. ✓

Corollary: Earliest finishing time algorithm is optimal.

Weighted Interval Scheduling

Weighted version of the problem:

- Each interval has a weight
- Goal: Non-overlapping set with maximum total weight

Earliest finishing time greedy algorithm fails:

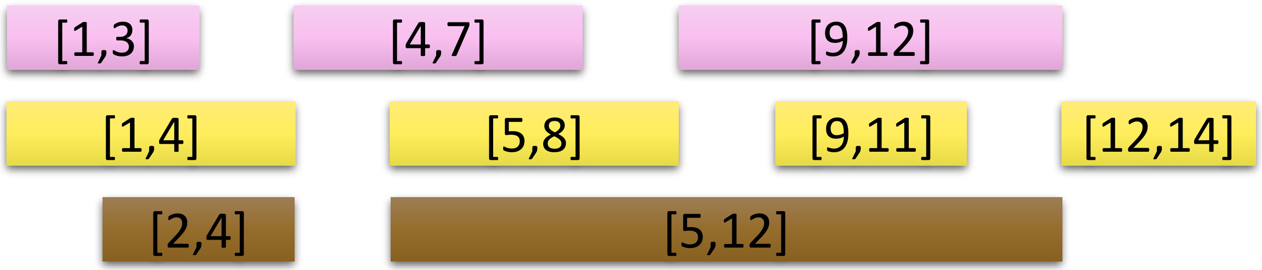
- Algorithm needs to look at weights
- Else, the selected sets could be the ones with smallest weight…

No simple greedy algorithm:

- We will see an algorithm using another design technique later.

Interval Partitioning

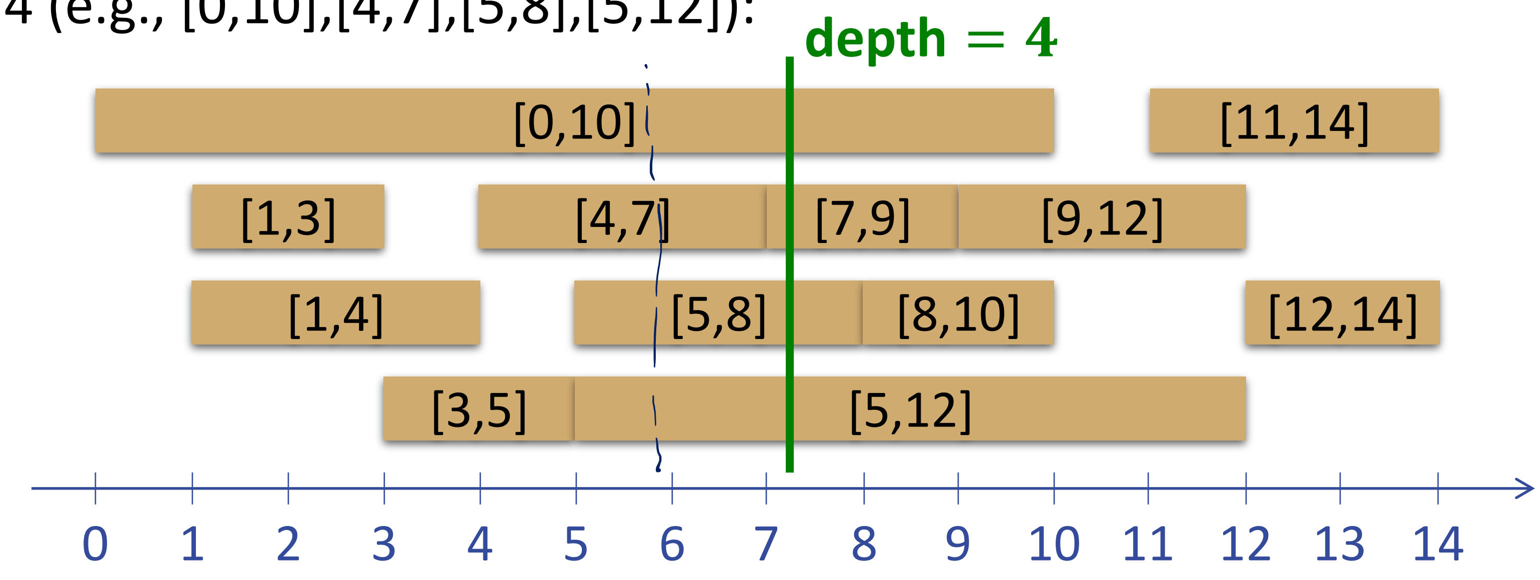
- **Schedule all intervals**: Partition intervals into as few as possible non-overlapping sets of intervals
  - Assign intervals to different resources, where each resource needs to get a non-overlapping set
- Example:
  - Intervals are requests to use some room during this time
  - Assign all requests to some room such that there are no conflicts
  - Use as few rooms as possible
- Assignment to 3 resources:



Depth

Depth of a set of intervals:

- Maximum number passing over a single point in time
  - Because we allow intervals to overlap at the boundaries, “passing” means in the inside of the interval.
- Depth of initial example is 4 (e.g., [0,10],[4,7],[5,8],[5,12]):



Lemma: Number of resources needed $\geq$ depth

- Follows directly from definition of depth.

Greedy Algorithm

Can we achieve a partition into "depth" non-overlapping sets?

- Would mean that the only obstacles to partitioning are local…

Algorithm:

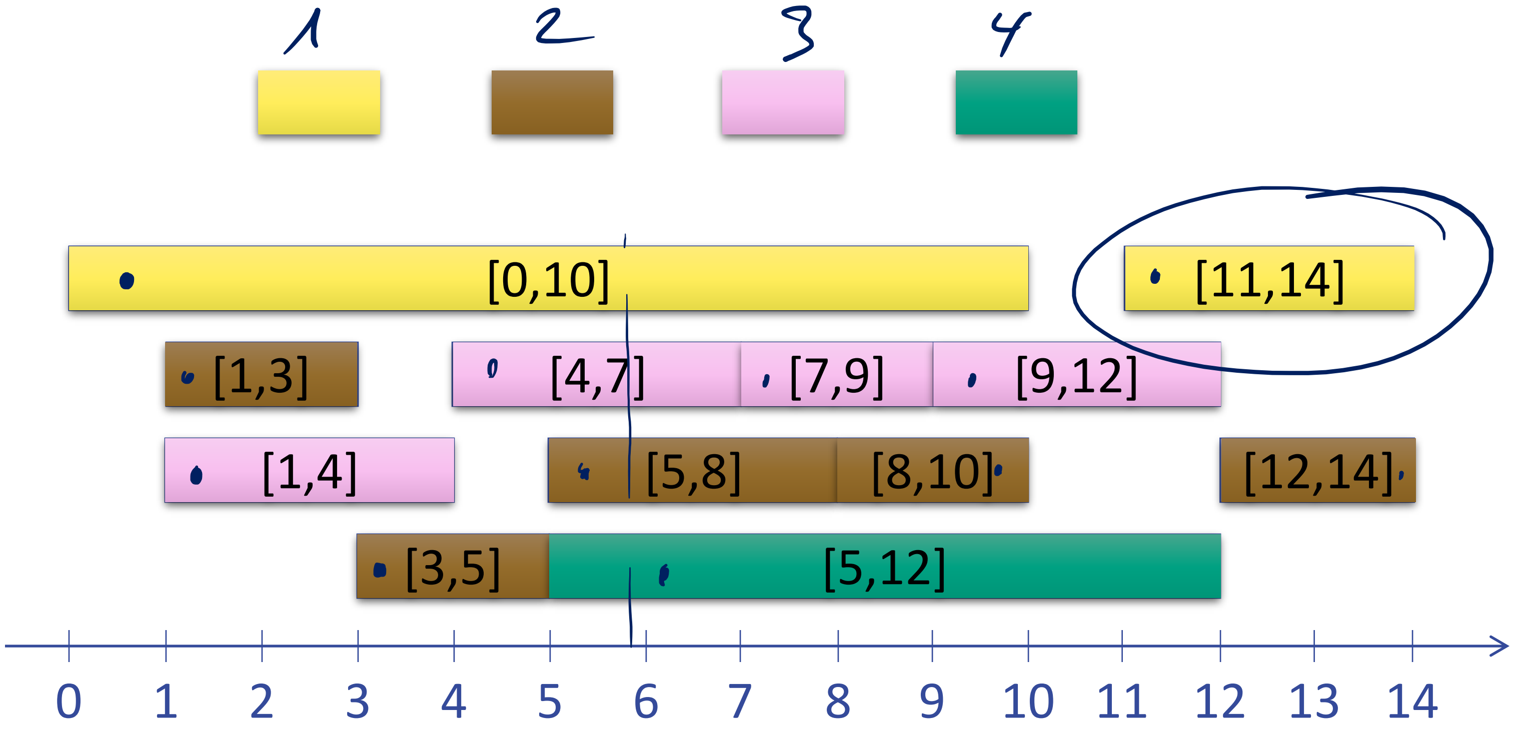
- Assign labels $1, \dots$ to the intervals; same label → non-overlapping

1. sort intervals by starting time: $I_1, I_2, \dots, I_n$
2. **for** $i = 1$ **to** n **do**
3. assign smallest possible label to I_i
   (possible label: different from conflicting intervals $I_j, j < i$)
4. **end**

Interval Partitioning Algorithm

Example:

- Labels:



- Number of labels = depth = 4

Interval Partitioning: Analysis

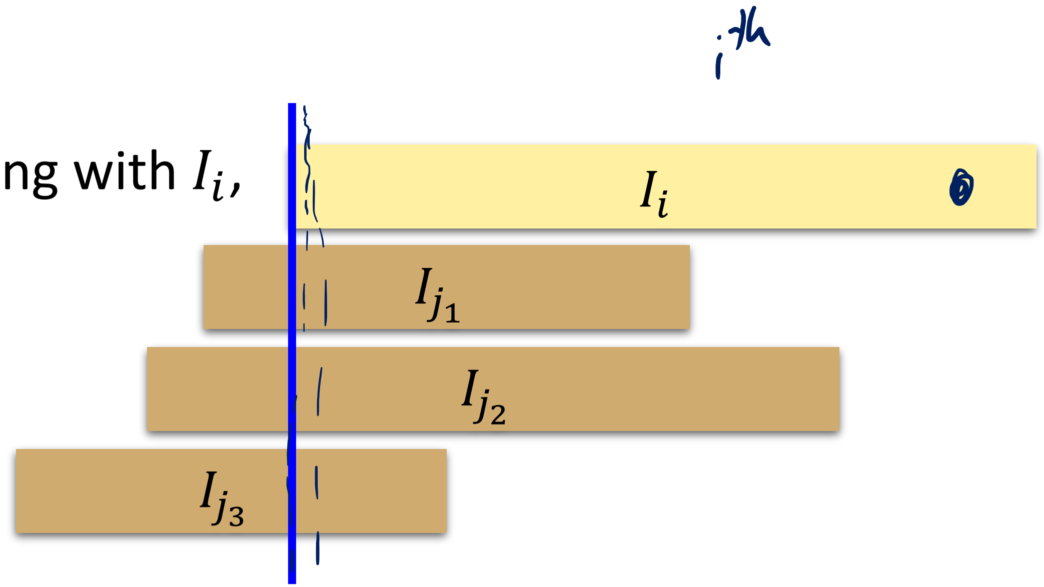
Theorem:

a) Let d be the depth of the given set of intervals.
The algorithm assigns a label from $1, \dots, d$ to each interval.

b) Sets with the same label are non-overlapping

Proof:

- b) holds by construction
- For a): All intervals $I_j, j < i$ overlapping with I_i,
  overlap at the beginning of I_i



  - At most $d - 1$ such intervals → some label in $\{1, \dots, d\}$ is available.

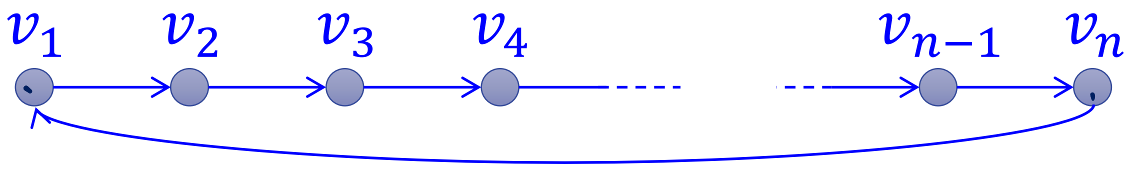
Traveling Salesperson Problem (TSP)

Input:

- Set V of n nodes (points, cities, locations, sites)
- Distance function $d: V \times V \rightarrow \mathbb{R}$, i.e., $d(u, v)$: dist. from u to v
- Distances usually symmetric, asymm. distances → asymm. TSP

Solution:

- Ordering/permutation $v_1, v_2, \dots, v_n$ of nodes:

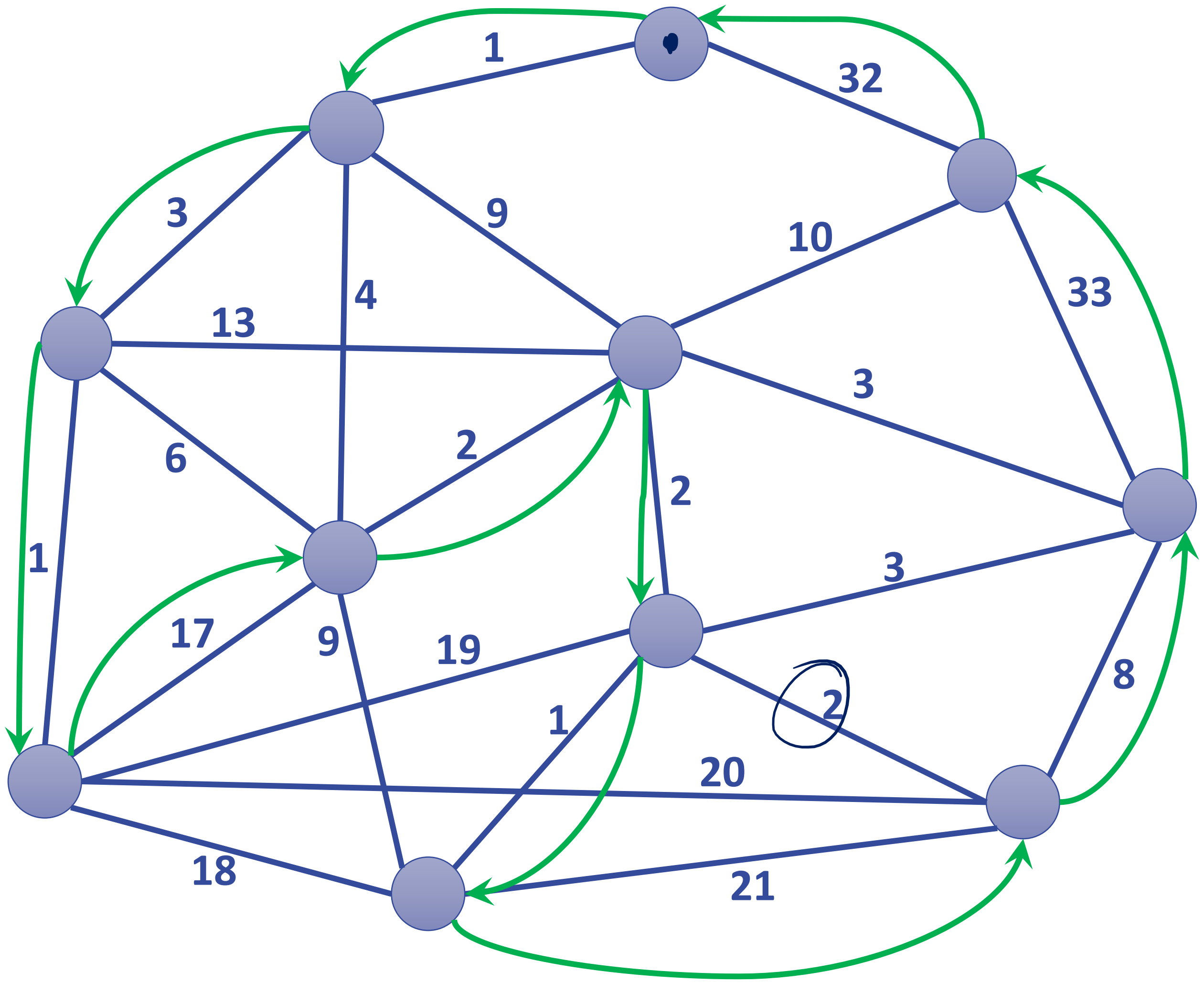


- Length of TSP path: $\sum_{i=1}^{n-1} d(v_i, v_{i+1})$
- Length of TSP tour: $d(v_n, v_1) + \sum_{i=1}^{n-1} d(v_i, v_{i+1})$

Goal:

- Minimize length of TSP path or TSP tour

Example



Greedy Algorithm

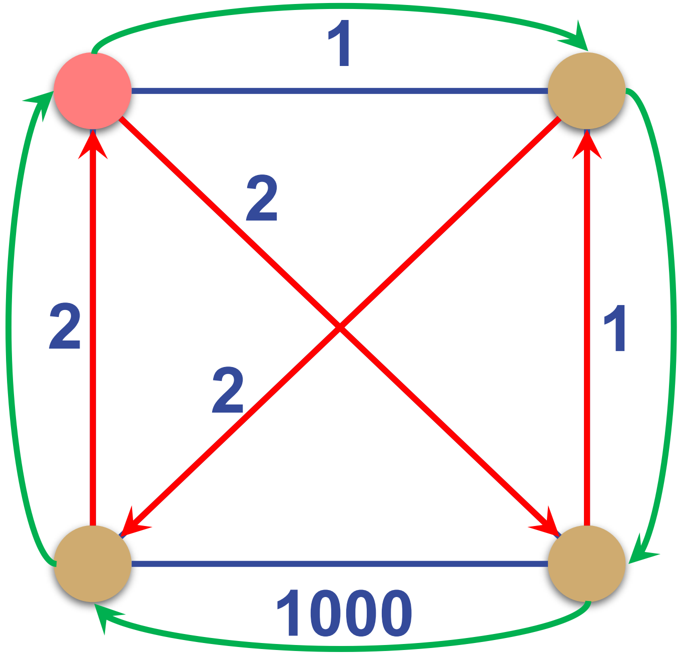
Length: 121

Optimal Tour

Length: 86

Nearest Neighbor (Greedy)

- Nearest neighbor can be arbitrarily bad, even for TSP paths

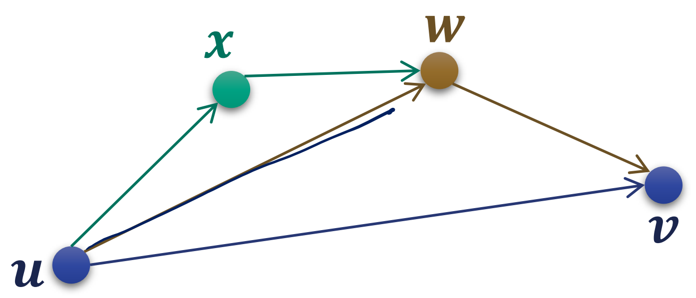


TSP Variants

- Asymmetric TSP
  - arbitrary non-negative distance function
  - most general, nearest neighbor arbitrarily bad
  - NP-hard to get within any bound of optimum

- Symmetric TSP
  - arbitrary non-negative symmetric distance function
  - nearest neighbor arbitrarily bad
  - NP-hard to get within any bound of optimum

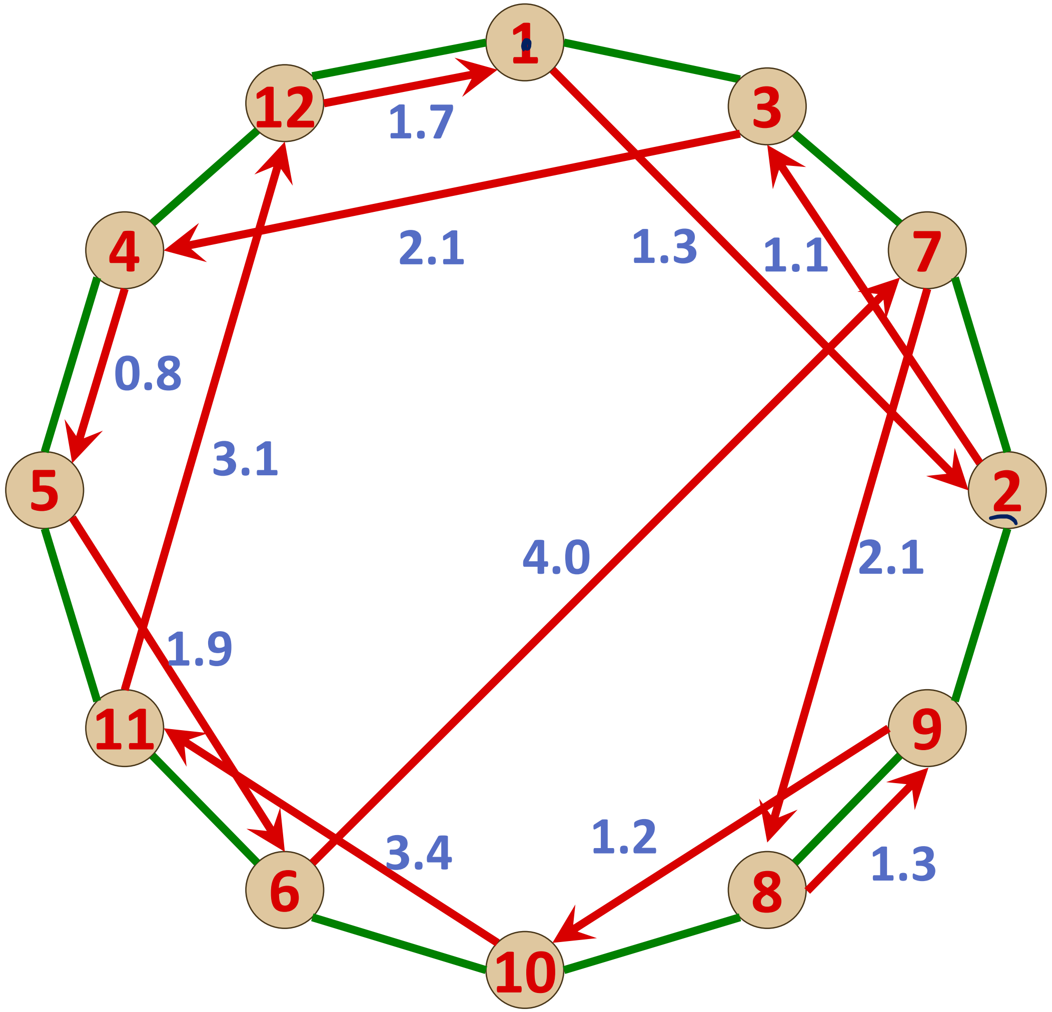
- Metric TSP
  - distance function defines metric space: symmetric, non-negative, triangle inequality: $d(u,v) \leq d(u,w) + d(w,v)$
  - possible to get close to optimum (we will later see how to get a factor ${}^{3}/_{2}$)
  - what about the nearest neighbor algorithm?



Metric TSP, Nearest Neighbor

Optimal TSP tour:

Nearest-Neighbor TSP tour:



Metric TSP, Nearest Neighbor

Optimal TSP tour:

Nearest-Neighbor TSP tour:

cost = 24

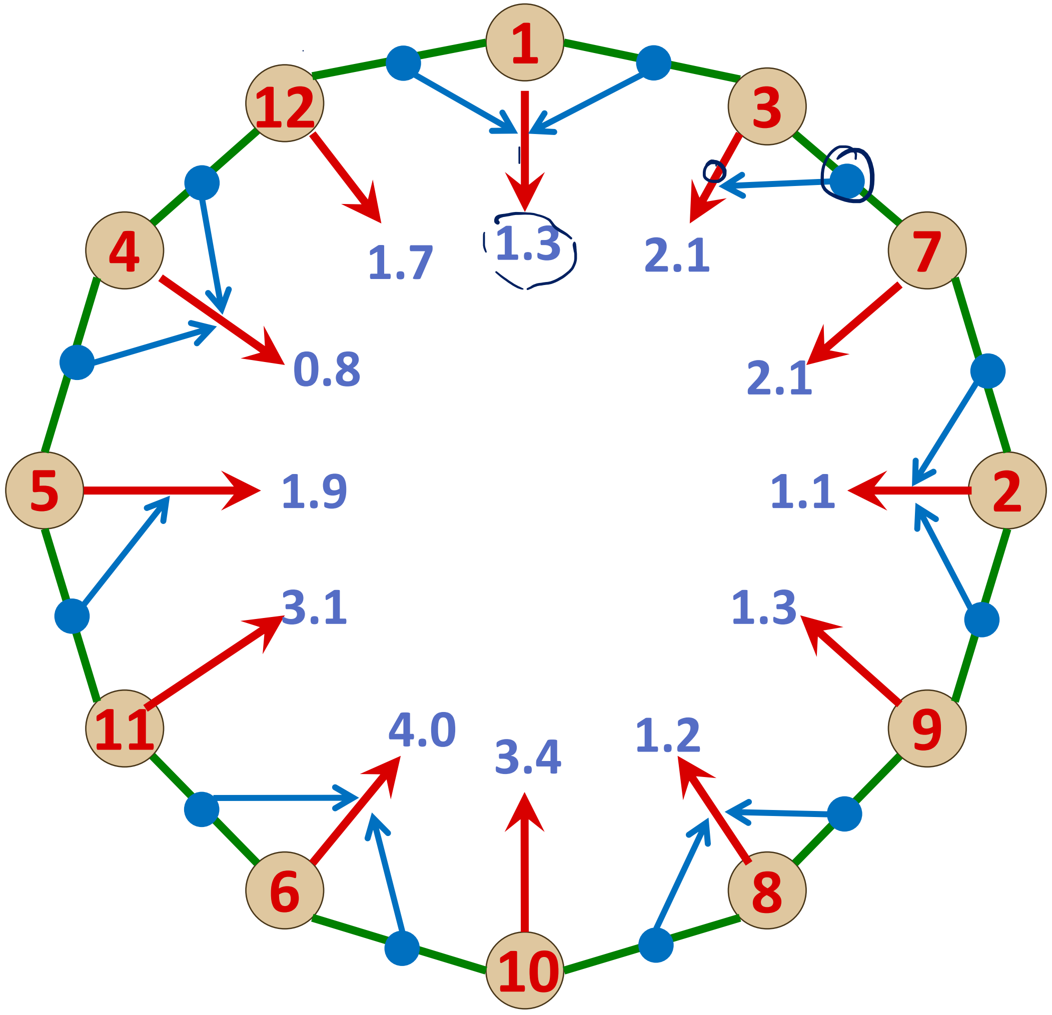
marked red edges : some arrow to it

green edges $\geq$ marked red ed.

OPT **part of greedy solution (NN)**

#marked red edges:

At least half of the red edges are marked.



Metric TSP, Nearest Neighbor

Triangle Inequality:

optimal tour on remaining nodes
(shortcut tour)

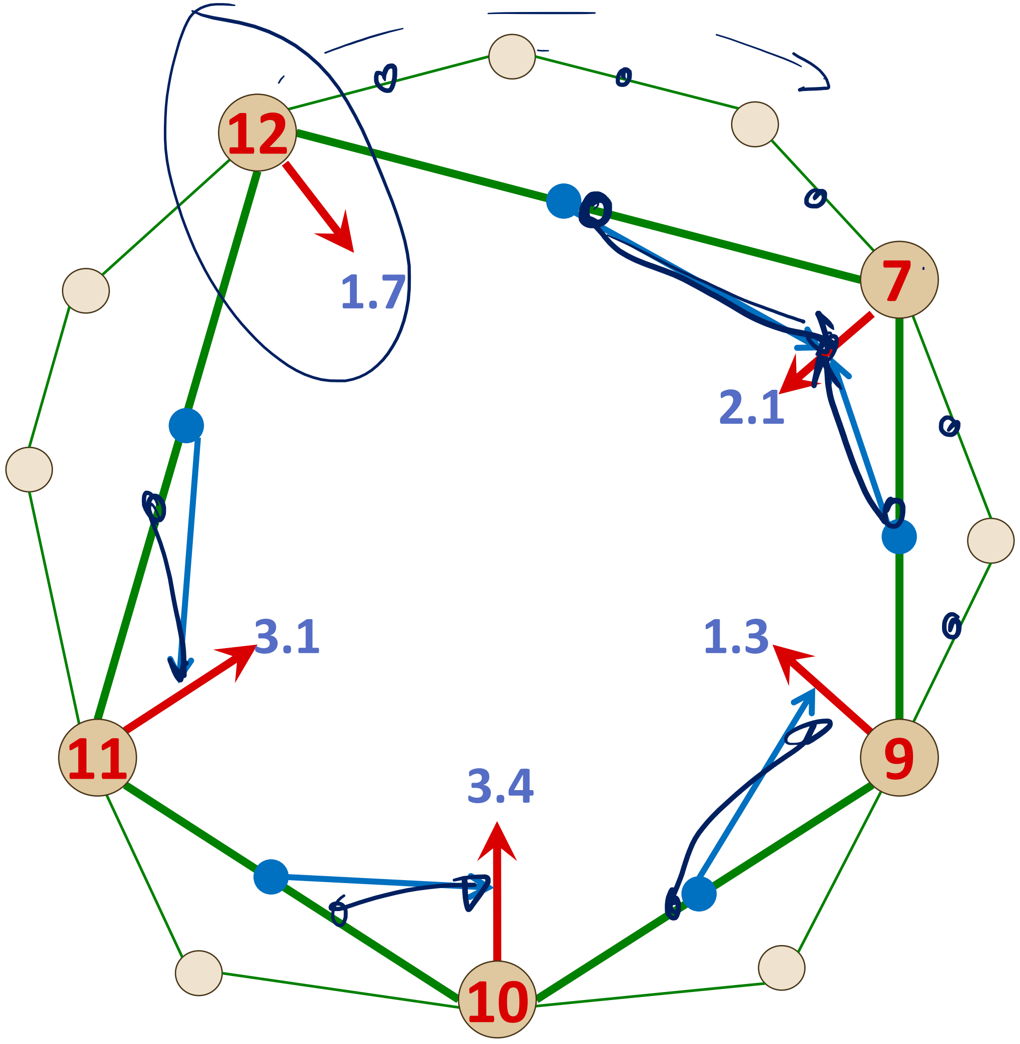
$\leq$

overall optimal tour

green $\geq$ marked red

$\leq$OPT

marked red $\leq$ OPT



Metric TSP, Nearest Neighbor

Analysis works in phases:

- In each phase, assign each optimal edge to some greedy edge
  - Cost of greedy edge $\leq$ cost of optimal edge
- Each greedy edge gets assigned ≤ 2 optimal edges
  - At least half of the greedy edges get assigned
- At end of phase:

  Remove nodes for which greedy edge is assigned

  Consider optimal solution for remaining points
- **Triangle inequality:** remaining opt. solution $\leq$ overall opt. sol.

- Cost of greedy edges assigned in **each phase $\leq$ opt. cost**
- **Number of phases $\leq \log_2 n$**
  - +1 for last greedy edge in tour

Metric TSP, Nearest Neighbor

- Assume:

  NN: cost of greedy tour, OPT: cost of optimal tour

- We have shown:

last red edge #phases

$$\frac{\text{NN}}{\text{OPT}} \leq \underbrace{1 + \log_2 n}_{\text{approximation ratio}}$$

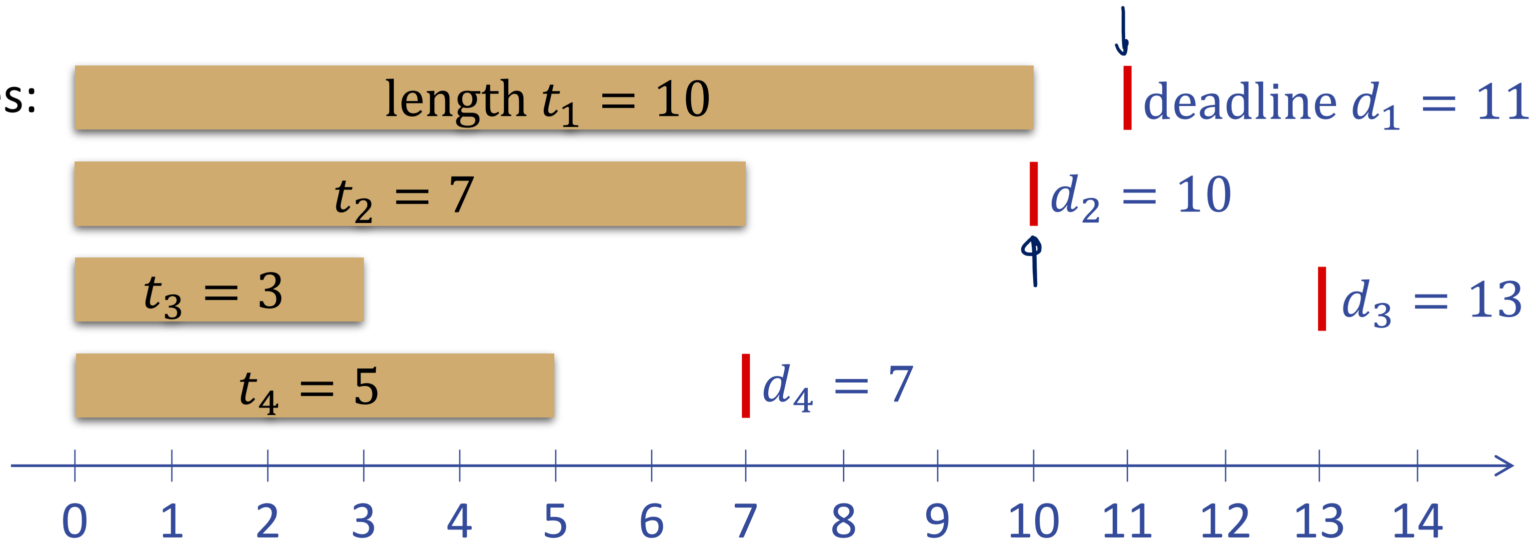
$$(\text{NN} \leq (1 + \log_2 n) \cdot \text{OPT})$$

- Example of an **approximation algorithm**

- We will later see a $^3/_2$-approximation algorithm for metric TSP

Back to Scheduling

- Given: n requests / jobs with deadlines:

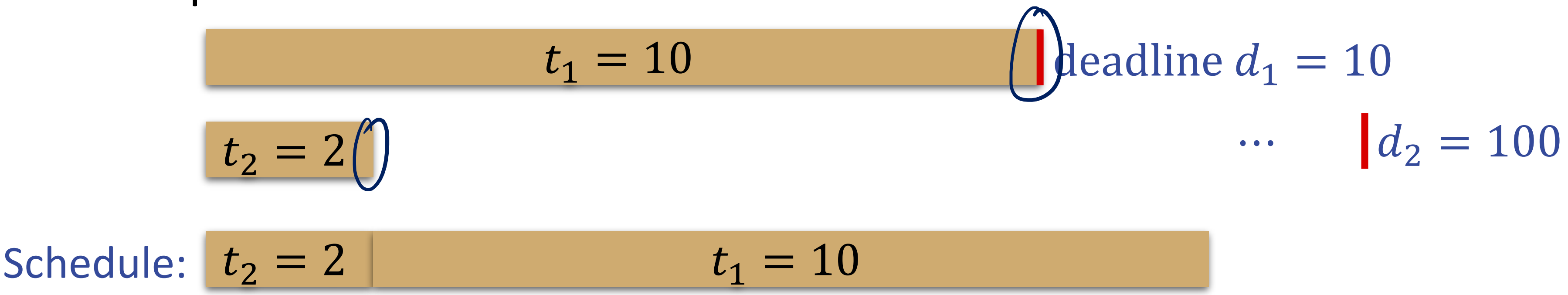


- Goal: schedule all jobs with minimum lateness L
  - Schedule: $s(i), f(i)$: start and finishing times of request i
    Note: $f(i) = s(i) + t_i$
  - Lateness L_i of request i : $L_i \coloneqq \max\{0, f(i) - d_i\}$
- Lateness $L \coloneqq \max\left\{0, \max_i\{f(i) - d_i\}\right\} = \max_i L_i$
  - largest amount of time by which some job finishes late
- Many other natural objective functions possible…

Greedy Algorithm?

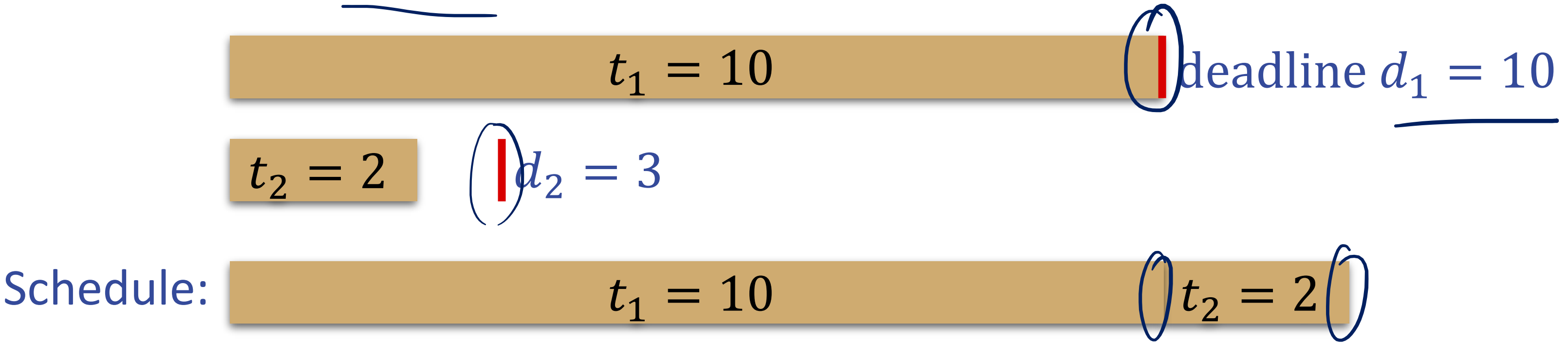
Schedule jobs in order of increasing length?

- Ignores deadlines: seems too simplistic…
- E.g.:



Schedule by increasing slack time?

- Should be concerned about slack time: $d_i - t_i$



Greedy Algorithm

Schedule by earliest deadline?

- Schedule in increasing order of d_i
- Ignores lengths of jobs: too simplistic?

- Earliest deadline is optimal!

Algorithm:

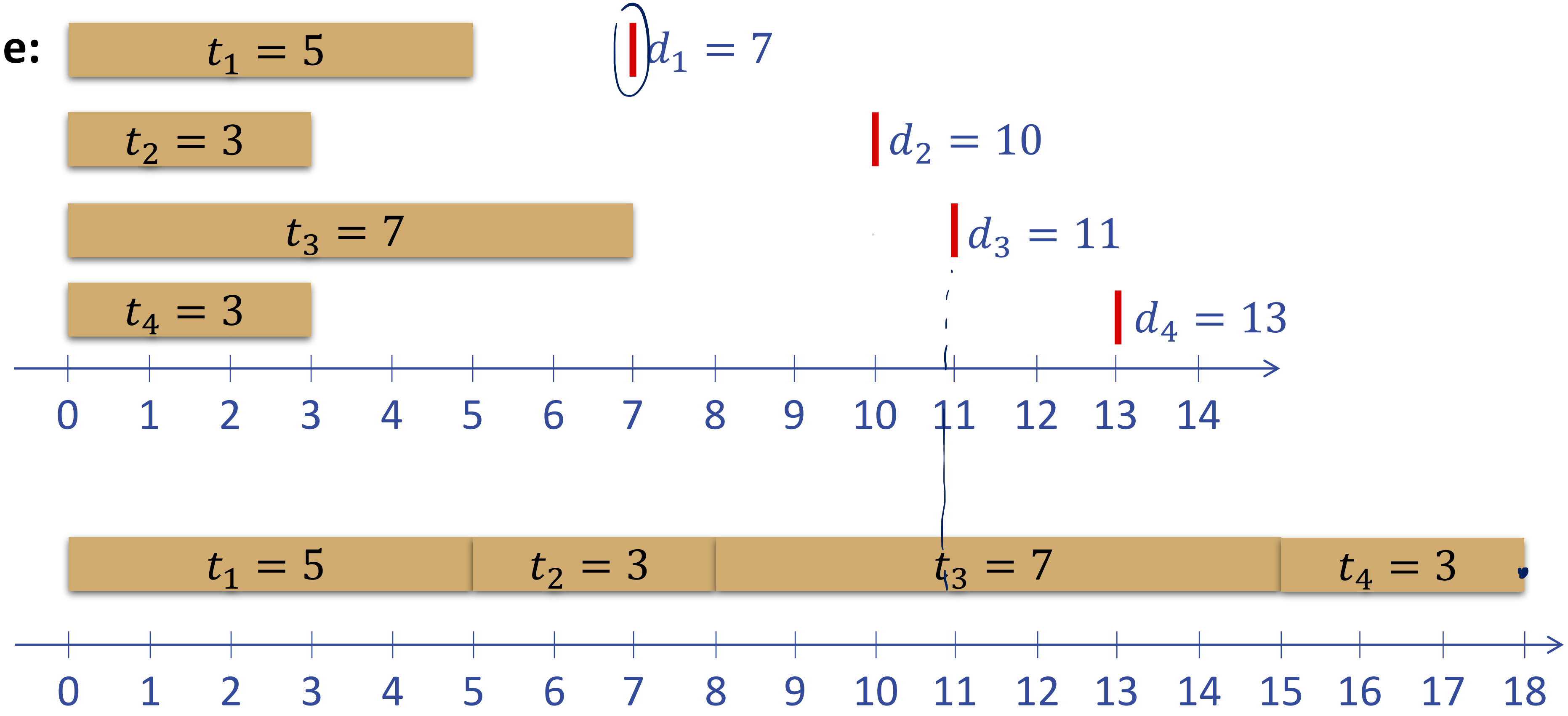
- Assume jobs are reordered such that $d_1 \leq d_2 \leq \cdots \leq d_n$
- Start/finishing times:
  - First job starts at time $s(1) = 0$
  - Duration of job i is t_i: $f(i) = s(i) + t_i$
  - No gaps between jobs: $s(i+1) = f(i)$

  (idle time: gaps in a schedule → alg. gives schedule with no idle time)

Example

Jobs ordered by deadline:

Schedule:

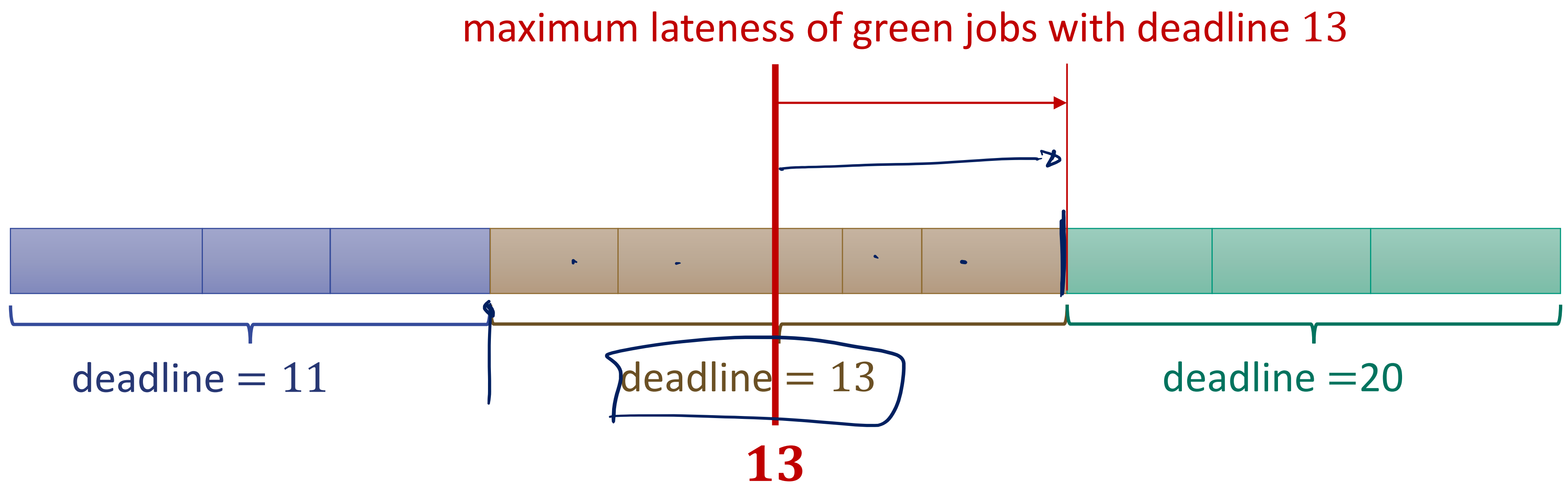


Lateness: job 1: 0, job 2: 0, job 3: 4, job 4: 5

Basic Facts

1. There is an optimal schedule with no idle time
   - Can just schedule jobs earlier…

2. Inversion: Job i scheduled before job j and $d_i > d_j$
   Schedules with no inversions have the same maximum lateness

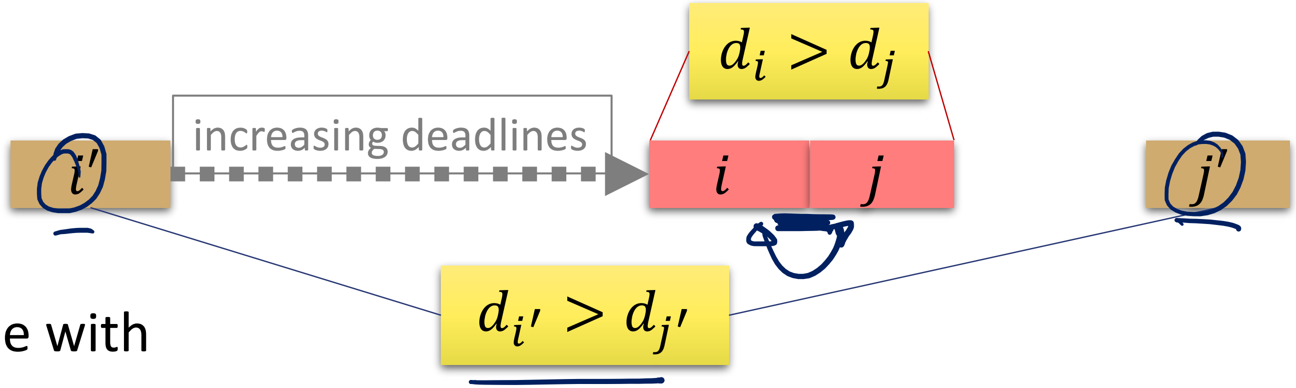


Earliest Deadline is Optimal

Theorem:
There is an optimal schedule $\mathcal{O}$ with no inversions and no idle time.

Proof:

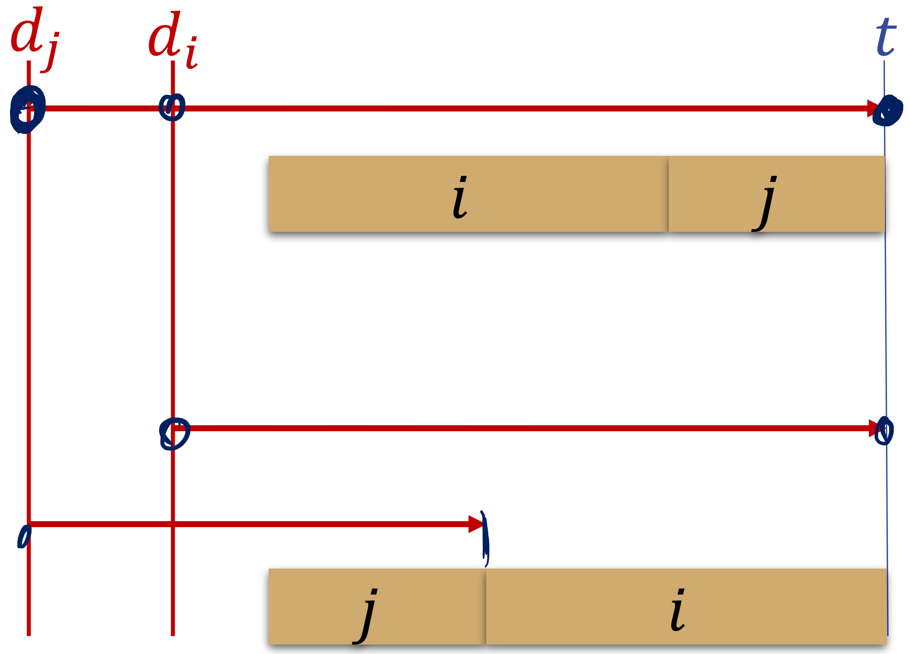
- Consider some schedule $\mathcal{O}'$ with no idle time
- If $\mathcal{O}'$ has inversions, ∃ pair (i, j), s.t. i is scheduled immediately before j and $d_j < d_i$



- Claim: Swapping i and j gives a schedule with
  1. Fewer inversions
  2. Maximum lateness no larger than in $\mathcal{O}'$

Earliest Deadline is Optimal

Claim: Swapping i and j: maximum lateness no larger than in $\mathcal{O}'$



Lateness $L_j = \max\{0, t - d_j\}$

Max. lateness after swap:

$L_i' = \max\{0, t - d_i\} \leq L_j$

$L_j' = \max\{0, L_j - t_i\} \leq L_j$

Exchange Argument

- General approach that often works to analyze greedy algorithms

- Start with any solution
- Define basic exchange step that allows to transform solution into a new solution that is not worse
- Show that exchange step move solution closer to the solution produced by the greedy algorithm
- Number of exchange steps to reach greedy solution should be finite…