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# Theoretical Computer Science - Bridging Course
# Exercise Sheet 7

**Due:** Tuesday, 10th of December 2024, 12:00 pm

## Exercise 1: Undecidable or Not Turing recongnizable Problems *(4+4 Points)*

1. Show that $EQ_{TM} = \{\langle M_1, M_2\rangle \mid M_1, M_2 \text{ are Turing Machines and } L(M_1) = L(M_2)\}$ is undecidable.

   *Hint: You may use that $E_{TM} = \{\langle M\rangle \mid M \text{ is a Turing Machine and } L(M) = \emptyset\}$ is undecidable.*

2. Fix an enumeration of all Turing machines (that have input alphabet $\Sigma$): $\langle M_1\rangle, \langle M_2\rangle, \langle M_3\rangle, \dots$.

   Fix also an enumeration of all words over $\Sigma$: $w_1, w_2, w_3, \dots$.

   Prove that language $L = \{w \in \Sigma^* \mid w = w_i, \text{ for some } i, \text{ and } M_i \text{ does not accept } w_i\}$ is not Turing recognizable.

   *Hint: Try to find a contradiction to the existence of a Turing machine that recognizes $L$.*

## Exercise 2: The Halting Problem Revisited *(4+4 Points)*

Show that both the halting problem and its special version are both undecidable.

1. The *halting problem* is defined as

   $$H = \{\langle M, w\rangle \mid \langle M\rangle \text{ encodes a TM and } M \text{ halts on string } w\}.$$

   *Hint: Assume $H$ is decidable and try to reach a contradiction by showing that some known undecidable problem (cf. from the lecture) is decidable.*

2. The *special halting problem* is defined as

   $$H_s = \{\langle M\rangle \mid \langle M\rangle \text{ encodes a TM and } M \text{ halts on } \langle M\rangle\}.$$

   *Hint: Assume that $M$ is a TM which decides $H_s$ and then construct a TM which halts iff $M$ does not halt. Use this construction to find a contradiction.*

## Exercise 3: $\mathcal{O}$-Notation Formal Proofs *(1+2+3 Points)*

Roughly speaking, the set $\mathcal{O}(f)$ contains all functions that are not growing faster than the function $f$ when additive or multiplicative constants are neglected. Formally:

$$g \in \mathcal{O}(f) \iff \exists c > 0, \exists M \in \mathbb{N}, \forall n \geq M : g(n) \leq c \cdot f(n)$$

For the following pairs of functions, state whether $f \in \mathcal{O}(g)$ or $g \in \mathcal{O}(f)$ or both. Proof your claims (you do not have to prove a negative result $\notin$, though).

(a) $f(n) = 100n,\ g(n) = 0.1 \cdot n^2$

(b) $f(n) = \sqrt[3]{n^2},\ g(n) = \sqrt{n}$

(c) $f(n) = \log_2(2^n \cdot n^3),\ g(n) = 3n$ *Hint: You may use that $\log_2 n \leq n$ for all $n \in \mathbb{N}$.*

## Exercise 4: Sort Functions by Asymptotic Growth *(7 Points)*

Give a sequence of the following functions sorted by asymptotic growth, i.e., for consecutive functions $g, f$ in your sequence, it should hold $g \in \mathcal{O}(f)$. Write "$g \cong f$" if $f \in \mathcal{O}(g)$ *and* $g \in \mathcal{O}(f)$.

| | | | |
|---|---|---|---|
| $\log_2(n!)$ | $\sqrt{n}$ | $2^n$ | $\log_2(n^2)$ |
| $3^n$ | $n^{100}$ | $\log_2(\sqrt{n})$ | $(\log_2 n)^2$ |
| $\log_{10} n$ | $10^{100} \cdot n$ | $n!$ | $n \log_2 n$ |
| $n \cdot 2^n$ | $n^n$ | $\sqrt{\log_2 n}$ | $n^2$ |