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Theoretical Computer Science - Bridging Course
Exercise Sheet 9

Due: Tuesday, 7th of January 2025, 12:00 pm

Exercise 1: Class $\mathcal{NPC}$ part 1

Let L_1, L_2 be languages (problems) over alphabets Σ_1, Σ_2. Then $L_1 \leq_p L_2$ (L_1 is polynomially reducible to L_2), iff a function $f : \Sigma_1^* \to \Sigma_2^*$ exists, that can be calculated in polynomial time and

$$\forall s \in \Sigma_1^* : s \in L_1 \iff f(s) \in L_2.$$

Language L is called $\mathcal{NP}$-hard, if *all* languages $L' \in \mathcal{NP}$ are polynomially reducible to L, i.e.

$$L \text{ is } \mathcal{NP}\text{-hard} \iff \forall L' \in \mathcal{NP} : L' \leq_p L.$$

The reduction relation '$\leq_p$' is transitive ($L_1 \leq_p L_2$ and $L_2 \leq_p L_3 \Rightarrow L_1 \leq_p L_3$). Therefore, in order to show that L is $\mathcal{NP}$-hard, it suffices to reduce a known $\mathcal{NP}$-hard problem $\tilde{L}$ to L, i.e. $\tilde{L} \leq_p L$.
Finally a language is called $\mathcal{NP}$-complete ($\Leftrightarrow$: $L \in \mathcal{NPC}$), if

1. $L \in \mathcal{NP}$ and
2. L is $\mathcal{NP}$-hard.

(a) Show $\textsc{Clique} := \{\langle G, k\rangle \,|\, G$ has a clique of size at least k $\} \in \mathcal{NPC}$.

(b) Show $\textsc{HittingSet} := \{\langle \mathcal{U}, S, k\rangle \mid$ universe $\mathcal{U}$ has subset of size at most k that *hits* all sets in $S \subseteq 2^{\mathcal{U}}\} \in \mathcal{NPC}$.

For both parts, use the fact that $\textsc{VertexCover} := \{\langle G, k\rangle \mid$ Graph G has a *vertex cover* of size at most $k\} \in \mathcal{NPC}$.

Hint: For the poly. transformation ($\leq_p$) you have to describe an algorithm (with poly. run-time!) that transforms:
for part (a), an instance $\langle G, k\rangle$ of VertexCover *into an instance $\langle G', k'\rangle$ of* Clique *s.t. a vertex cover of size $\leq k$ in G becomes a clique of G' of size $\geq k'$ vice versa(!)*
for part (b), an instance $\langle G, k\rangle$ of VertexCover *into an instance $\langle \mathcal{U}, S, k'\rangle$ of* HittingSet*, s.t. a vertex cover of size $\leq k$ in G becomes a hitting set of $\mathcal{U}$ of size $\leq k'$ for S and vice versa(!).*

Exercise 2: Class $\mathcal{NPC}$ part 2

1. Given a set U of n elements ('universe') and a collection $S \subseteq \mathcal{P}(U)$ of subsets of U, a selection $C_1, \ldots, C_k \in S$ of k sets is called a *set cover* of (U, S) of size k if $C_1 \cup \ldots \cup C_k = U$.

   Show that the problem

   $$\textsc{SetCover} := \{\langle U, S, k\rangle \,|\, U \text{ is a set, } S \subseteq \mathcal{P}(U) \text{ and there is a set cover of } (U, S) \text{ of size } k\}$$

   is NP-complete.

   You may use that

   $$\textsc{DominatingSet} = \{\langle G, k\rangle \mid G \text{ has a dominating set with } k \text{ nodes}\}.$$

   is NP-complete.

2. Show $\textsc{DominatingSet} := \{\langle G, k\rangle \mid \text{Graph } G \text{ has a dominating set of size at most } k\} \in \mathcal{NPC}$.

   Use that $\textsc{VertexCover} := \{\langle G, k\rangle \mid \text{Graph } G \text{ has a vertex cover of size at most } k\} \in \mathcal{NPC}$.

   *Remark: A **vertex cover** is a subset of nodes of G such that every edge of G is incident to a node in the subset.*

   Hint: Transform a Graph G into a Graph G′ such that a vertex cover of G will result in a dominating set G′ and vice versa(!). Note that a dominating set is not necessarily a vertex cover ($G = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\})$*) has the dominating set* $\{v_1, v_4\}$ *which is not a vertex cover). Also a vertex cover is not necessarily a dominating set (consider isolated notes).*