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Theoretical Computer Science - Bridging Course
Exercise Sheet 10

Due: Tuesday, 14th of January 2025, 12:00 pm

Exercise 1: Propositional Logic: Basic Terms *(1+1+1+1 Points)*

Let $\Sigma := \{p, q, r\}$ be a set of atoms. An interpretation $I : \Sigma \to \{T, F\}$ maps every atom to either true or false. Inductively, an interpretation I can be extended to composite formulae φ over Σ (cf. lecture). We write $I \models \varphi$ if φ evaluates to T (true) under I. In case $I \models \varphi$, I is called a *model* for φ.

For each of the following formulae, give *all* interpretations which are models. Make a truth table and/or use logical equivalencies to find all models (document your steps). Which of these formulae are satisfiable, which are unsatisfiable and which are tautologies?

(a) $\varphi_1 = (p \land \neg q) \lor (\neg p \lor q)$

(b) $\varphi_2 = (\neg p \land (\neg p \lor q)) \leftrightarrow (p \lor \neg q)$

(c) $\varphi_3 = (p \land \neg q) \to \neg(p \land q)$

(d) $\varphi_4 = (p \land q) \to (p \lor r)$

Remark: $a \to b :\equiv \neg a \lor b$, $a \leftrightarrow b :\equiv (a \to b) \land (b \to a)$, $a \not\to b :\equiv \neg(a \to b)$.

Exercise 2: CNF and DNF *(2+1 Points)*

(a) Convert $\varphi_1 := (p \to q) \to (\neg r \land q)$ into Conjunctive Normal Form (CNF).

(b) Convert $\varphi_2 := \neg((\neg p \to \neg q) \land \neg r)$ into Disjunctive Normal Form (DNF).

Remark: Use the known logical equivalencies given in the lecture slides to do the necessary transformations. State which equivalency you are using in each step.

Exercise 3: Logical Entailment *(2+2 Points)*

A *knowledge base* KB is a set of formulae over a given set of atoms Σ. An interpretation I of Σ is called a model of KB, if it is a model for *all* formulae in KB. A knowledge base KB *entails* a formula φ (we write $KB \models \varphi$), if *all* models of KB are also models of φ.

Let $KB := \{p \lor q, \neg r \lor p\}$. Show or disprove that KB logically entails the following formulae.

(a) $\varphi_1 := (p \land q) \lor \neg(\neg r \lor p)$

(b) $\varphi_2 := (q \leftrightarrow r) \to p$

Exercise 4: Inference Rules and Calculi *(2+2 Points)*

Let $\varphi_1, \ldots, \varphi_n, \psi$ be propositional formulae. An *inference rule*

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}$$

means that if $\varphi_1, \ldots, \varphi_n$ are 'considered true', then ψ is 'considered true' as well ($n = 0$ is the special case of an axiom). A (propositional) *calculus* $\mathbf{C}$ is described by a *set* of inference rules.
Given a formula ψ and knowledge base $KB := \{\varphi_1, \ldots, \varphi_n\}$ (where $\varphi_1, \ldots, \varphi_n$ are formulae) we write $KB \vdash_{\mathbf{C}} \psi$ if ψ can be derived from KB by starting from a subset of KB and repeatedly applying inference rules from the calculus $\mathbf{C}$ to 'generate' new formulae until ψ is obtained.

Consider the following two calculi, defined by their inference rules (φ, ψ, χ are arbitrary formulae).

$$\mathbf{C_1}: \quad \frac{\varphi \to \psi, \psi \to \chi}{\varphi \to \chi}, \frac{\neg\varphi \to \psi}{\neg\psi \to \varphi}, \frac{\varphi \leftrightarrow \psi}{\varphi \to \psi, \psi \to \varphi}$$

$$\mathbf{C_2}: \quad \frac{\varphi, \varphi \to \psi}{\psi}, \frac{\varphi \wedge \psi}{\varphi, \psi}, \frac{(\varphi \wedge \psi) \to \chi}{\varphi \to (\psi \to \chi)}$$

Using the respective calculus, show the following derivations (document your steps).

(a) $\{p \leftrightarrow \neg r, \neg q \to r\} \vdash_{\mathbf{C_1}} p \to q$

(b) $\{p \wedge q, p \to r, (q \wedge r) \to s\} \vdash_{\mathbf{C_2}} s$

Remark: Inferences of a given calculus are purely syntactical, i.e. rules only apply in their specific form (much like a grammar) and no other logical transformations not given in the calculus are allowed.

Exercise 5: Resolution Calculus *(1+1+3 Points)*

Due to the *Contradiction Theorem* (cf. lecture) for every knowledge base KB and formula φ it holds

$$KB \models \varphi \quad \Longleftrightarrow \quad KB \cup \{\neg\varphi\} \models \bot.$$

Remark: $\bot$ is a formula that is unsatisfiable.

In order to show that KB entails φ, we show that $KB \cup \{\neg\varphi\}$ entails a contradiction. A calculus $\mathbf{C}$ is called *refutation-complete* if for every knowledge base KB

$$KB \models \bot \quad \Longrightarrow \quad KB \vdash_{\mathbf{C}} \bot.$$

Hence, given a refutation-complete calculus $\mathbf{C}$ it suffices to show $KB \cup \{\neg\varphi\} \vdash_{\mathbf{C}} \bot$ to prove $KB \models \varphi$.

The *Resolution Calculus* $\mathbf{R}$ is a formal way to do a prove by contradiction. It is correct and refutation-complete[1] for knowledge bases that are given in *Conjunctive Normal Form* (CNF). A knowledge base KB is in CNF if it is of the form $KB = \{C_1, \ldots, C_n\}$ where its clauses $C_i = \{L_{i,1}, \ldots, L_{i,m_i}\}$ each consist of m_i literals $L_{i,j}$.
Remark: KB represents the formula $C_1 \wedge \ldots \wedge C_n$ with $C_i = L_{i,1} \vee \ldots \vee L_{i,m_i}$.

The Resolution Calculus has only one inference rule, the *resolution rule*:

$$\mathbf{R}: \quad \frac{C_1 \cup \{L\}, C_2 \cup \{\neg L\}}{C_1 \cup C_2}.$$

Remark: L is a literal and $C_1 \cup \{L\}, C_2 \cup \{\neg L\}$ are clauses in KB (C_1, C_2 may be empty). To show $KB \vdash_{\mathbf{R}} \bot$, you need to apply the resolution rule, until you obtain two conflicting one-literal clauses L and $\neg L$. These entail the empty clause (defined as $\Box$), i.e. a contradiction ($\{L\}, \{\neg L\} \vdash_{\mathbf{R}} \bot$).

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[1] Complete calculi are impractical, since they have too many inference rules. More inference rules make automated proving with a computer significantly more complex. The Resolution Calculus is an appropriate technique to avoid this additional complexity, since it has only one inference rule.

(a) We want to show $\{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow u\} \models u$. First convert this problem instance into a form that can be solved via resolution as described above. Document your steps.

(b) Now, use resolution to show $\{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow u\} \models u$.

(c) Consider the sentence "Heads, I win". "Tails, you lose". Design a propositional KB that represents these sentences (create the propositions and rules required). Then use propositional resolution to prove that **I always win**.