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Theoretical Computer Science - Bridging Course
Sample Solution Exercise Sheet 2

Due: Tuesday, 5th of November 2024, 12:00 pm

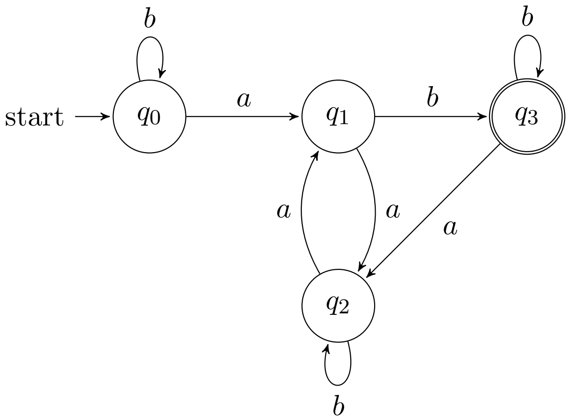
Exercise 1: Constructing DFAs, NFAs *(2+2+2 Points)*

Construct DFAs that recognize the first two languages and an NFA that recognizes the last language. The alphabet set is $\Sigma = \{a, b\}$.

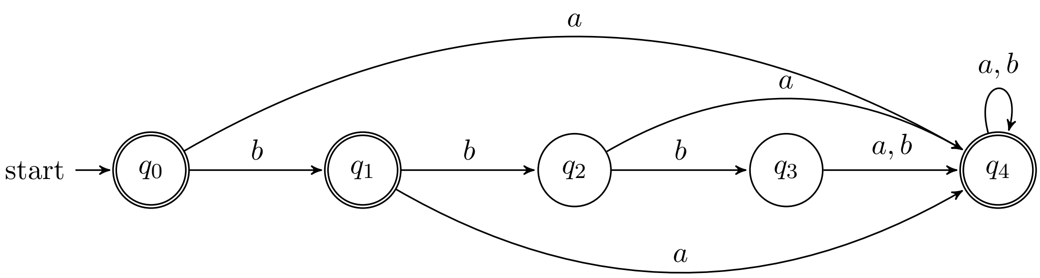
1. $L_1 = \{w \mid w$ has an odd number of a's and ends with $b\}$.
2. $L_2 = \{w \mid w$ is any string except bb and $bbb\}$.
3. $L_3 = \{w \mid w$ is any string where at least one of the symbols a or b occurs an even number of times$\}$.

Sample Solution

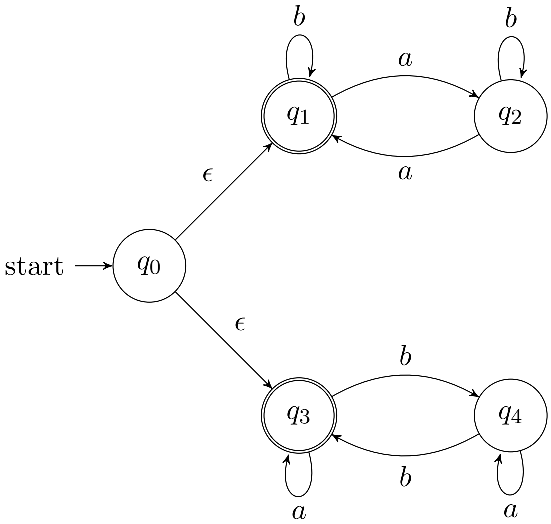
1.



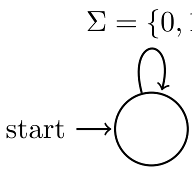
2.



3.



Bonus :
The following DFA accepts the empty language:



Remark: There's a difference between the following two languages $L_1 := \emptyset$ and $L_2 := \{\varepsilon\}$, where the empty string ε is defined as a string of length $|\varepsilon| = 0$.
The empty language $\emptyset$ is a *set* containing no strings, while $L_2 = \{\varepsilon\}$ is a *set* containing ε, while ε is just a *string* but a string containing no symbols. So, L_1, L_2 are different languages since L_2 contains a string while L_1 is empty $(0 = |L_1| \neq |L_2| = 1)$.

Exercise 2: Closure of Regular Languages *(2+3+2+2 Points)*

1. Show that if M is a DFA that recognizes language L, you can construct a new DFA M' that recognizes the complement of L i.e. $\bar{L} := \Sigma^* \setminus L$. Conclude that the class of regular languages is closed under complementation.
2. Show by giving an example that if M is an NFA (instead of a DFA) that recognizes language L, then the same approach you used to construct the new DFA M' above doesn't necessarily yield a new NFA that recognizes the complement of L. Is the class of languages recognized by NFAs closed under complementation? Explain your answer.

   Let L_1 and L_2 be regular languages.

3. Show that $L_1 \cap L_2$ is regular by constructing its corresponding DFA.
4. Deduce from parts 1 and 3 that regular languages are closed under the symmetric difference i.e. $L_1 \Delta L_2$ is also regular.

Remark: For parts 1 and 3 there's no need for drawing state diagrams. Show how a DFA for the language in the question can be constructed presuming the existence of DFAs for L, L_1, and L_2.

Sample Solution

1. Let $M = (Q, \Sigma, \delta, q_0, F)$ be the DFA recognizing L. We define the DFA $M' := (Q, \Sigma, \delta, q_0, \overline{F})$ by inverting the set of accepting states of M, i.e. $\overline{F} := Q \setminus F$. We show that M' recognizes $\overline{L}$.

   If $w \in \overline{L}$, then $w \notin L$ and so M halts in an non accepting state q when processing w. M' will halt in the same state (because we only changed the set of accepting states), but here q is an accepting state. Analogously, if $w \notin \overline{L}$, then $w \in L$ and so M halts in an accepting state when processing w. M' will again halt in the same state, but here q is a non accepting state. So we have that M' halts in an accepting state when processing w if and only if $w \in \overline{L}$.
   Therefore, if M is a DFA that recognizes language L (so L is a regular language), then we can construct a new DFA M' that recognizes the language $\overline{L}$ (so $\overline{L}$ is also regular). Hence, the class of regular languages is closed under complementation (notice that the class of languages recognized by DFAs is actually the class of regular languages (Definition 1.16 p. 19)).

2. Consider the NFA given in Exercise 3 below. Consider string "ab". It is accepted in the original NFA. (In particular, the NFA can halt in state q_2.) However, once you swap the non-accept and accept states, in the new NFA, this string is still accepted. (In particular, the new NFA can halt in state q_0.)
   Yes, since the class of languages recognized by NFAs is actually the class of regular languages (Corollary 1.40 p.47) and we have just proved, by the previous part, that regular languages are closed under complementation, hence the class of languages recognized by NFAs is closed under complementation.

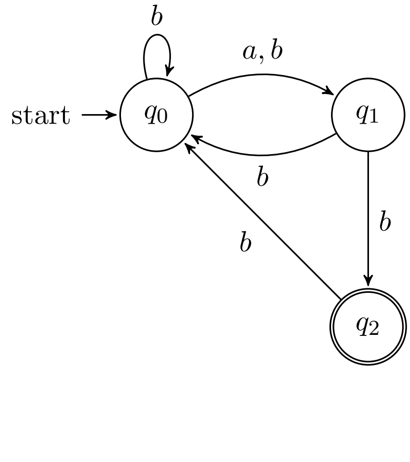
3. For proving the regularity of $L_1 \cap L_2$, we construct the product automaton like done in the lecture (Theorem 1.25. p. 30) for $L_1 \cup L_2$, with the difference that we set $F := F_1 \times F_2$ as the set of accepting states, where F_1 and F_2 are the sets of accepting states of the DFAs for L_1 and L_2.

   Alternative approach: using *De Morgan's law* we obtain: $L_1 \cap L_2 = \overline{(\overline{L_1} \cup \overline{L_2})}$. Thus $L_1 \cap L_2$ is regular, since we already know that regularity is conserved by complementation and a finite number of unions of regular languages (cf. lecture).

4. We know that the set difference of languages L_1 and L_2 is defined as $L_1 \backslash L_2 = L_1 \cap \overline{L_2}$ and after showing that regular languages are closed under intersection and complement in parts 1 and 3 respectively, it follows that regular languages are also closed under set difference.
   Finally, we have that the symmetric difference of L_1 and L_2 is defined as $L_1 \Delta L_2 = (L_1 \setminus L_2) \cup (L_2 \setminus L_1)$ and we know from the lecture that regular languages are closed under union. Moreover, we have just proved that regular languages are also closed under set difference, hence regular languages are also closed under the symmetric difference.

Exercise 3: NFA to DFA *(2+3 Points)*

Consider the following NFA.



1. Give a formal description of the NFA by giving the alphabet, state set, transition function, start state and the set of accept states.
2. Construct a DFA which is equivalent to the above NFA by drawing the corresponding state diagram.

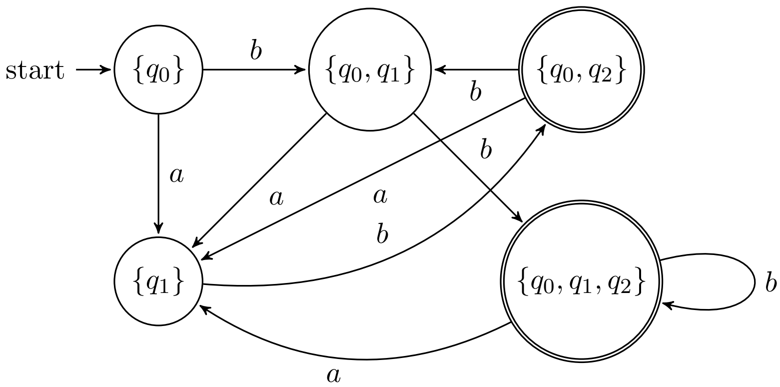
Bonus question: Explain which language the automaton accepts.

Sample Solution

1. The set of states is $Q = \{q_0, q_1, q_2\}$; the alphabet $\Sigma = \{a, b\}$; the starting state is q_0; the set of accept states is $F = \{q_2\}$; the transition function is shown in the following table.

	q_0	q_1	q_2
a	$\{q_1\}$	$\emptyset$	$\emptyset$
b	$\{q_0, q_1\}$	$\{q_0, q_2\}$	$\{q_0\}$
ϵ	$\emptyset$	$\emptyset$	$\emptyset$

2. After performing the algorithm from the lecture, we obtain the following DFA. All transitions which are not in the picture go to the garbage state $\emptyset$.



Bonus solution: The recognized language contains words of length at least two. Furthermore any a is immediately followed by a 'b'. The number of b's after the last a must not be two.