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Theoretical Computer Science - Bridging Course
Sample Solution Exercise Sheet 6

Due: Tuesday, 3rd of December 2024, 12:00 pm

Exercise 1: Decidable Problems *(2+2 +2+2 Points)*

Show that the following languages are decidable.

1. $A = \{\langle R, S\rangle \mid R$ and S are regular expressions and $L(R) \subseteq L(S)\}$.
2. Let $B = \{\langle G\rangle \mid G$ is a CFG over $\{0,1\}$ and $1^* \cap L(G) \neq \phi\}$. Use the fact that a language $C \cap R$ is context free for some context free language C and regular language R.
3. Consider a decidable language L. Show that its complement $\bar{L}$ is also decidable.
4. Show that there is a language that is Turing-Recognizable but not Turing-Decidable using the result above.

Sample Solution

1. Let T be the Turing Machine deciding the language $\{\langle D\rangle \mid D$ is a DFA with $L(D) = \emptyset\}$ (known from the lecture). We have $L(R) \subseteq L(S) \Leftrightarrow L(R) \setminus L(S) = \emptyset$. Thus we construct a decider for A in the following way.
   A'= " On input $\langle R, S\rangle$ where R, S are regular expression:
   1. Convert R and S into equivalent DFAs (like in the lecture)
   2. Construct a DFA D for the regular language $L(R) \setminus L(S) = \overline{\overline{L(R)} \cup L(S)}$
   3. Run T on input $\langle D\rangle$. Accept iff T accepts."
2. From the the fact, we deduce that $1^* \cap L(G)$ is context free. We construct a TM that decides on B as follows.
   B' =" On input $< G >$, where G is a context free grammar :
   1. Construct a CFG C such that $L(C) = 1^* \cap L(G)$
   2. Test whether $L(C) = \phi$ using the decider R for E_{CFG} (cf. lecture).
   3. If R accepts, reject; if R rejects, accept."
3. As the language L is decidable, it has a Turing Machine M that halts on every input string w. So, it either accepts and halts, or rejects and halts. Using this, we can construct a TM M' that does the following :
   1. Run M on input w.
   2. If M accepts, M' rejects.
   3. If M rejects, M accepts.

   This TM M' not only recognizes the language L, but also is decidable, as it halts on every input.
4. Suppose there is a language L that is Turing-Recognizable. Then its complement L is not TuringRecognizable, and this means this language is also not Turing-Decidable, i.e. Undecidable. That derives the fact that this language L is also undecidable.