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Theoretical Computer Science - Bridging Course
Sample Solution Exercise Sheet 10

Due: Tuesday, 14th of January 2025, 12:00 pm

Exercise 1: Propositional Logic: Basic Terms *(1+1+1+1 Points)*

Let $\Sigma := \{p, q, r\}$ be a set of atoms. An interpretation $I : \Sigma \to \{T, F\}$ maps every atom to either true or false. Inductively, an interpretation I can be extended to composite formulae φ over Σ (cf. lecture). We write $I \models \varphi$ if φ evaluates to T (true) under I. In case $I \models \varphi$, I is called a *model* for φ.

For each of the following formulae, give *all* interpretations which are models. Make a truth table and/or use logical equivalencies to find all models (document your steps). Which of these formulae are satisfiable, which are unsatisfiable and which are tautologies?

(a) $\varphi_1 = (p \wedge \neg q) \vee (\neg p \vee q)$

(b) $\varphi_2 = (\neg p \wedge (\neg p \vee q)) \leftrightarrow (p \vee \neg q)$

(c) $\varphi_3 = (p \wedge \neg q) \to \neg(p \wedge q)$

(d) $\varphi_4 = (p \wedge q) \to (p \vee r)$

Remark: $a \to b :\equiv \neg a \vee b$, $a \leftrightarrow b :\equiv (a \to b) \wedge (b \to a)$, $a \not\to b :\equiv \neg(a \to b)$.

Sample Solution

(a) See Table 1. The result shows that φ_1 is a tautology.

(b) See Table 2. The result shows that φ_2 is satisfiable.

(c) $\varphi_3 \equiv \neg(p \wedge \neg q) \vee (\neg p \vee \neg q) \equiv (\neg p \vee q) \vee (\neg p \vee \neg q) \equiv \neg p \vee q \vee \neg p \vee \neg q \equiv \neg p \vee \neg q \vee q$ which is a tautology as either q or $\neg q$ holds.

(d) See Table 3. The result shows that φ_4 is a tautology.

Exercise 2: CNF and DNF *(2+1 Points)*

(a) Convert $\varphi_1 := (p \to q) \to (\neg r \wedge q)$ into Conjunctive Normal Form (CNF).

(b) Convert $\varphi_2 := \neg((\neg p \to \neg q) \wedge \neg r)$ into Disjunctive Normal Form (DNF).

Remark: Use the known logical equivalencies given in the lecture slides to do the necessary transformations. State which equivalency you are using in each step.

model	p	q	$p \wedge \neg q$	$\neg p \vee q$	φ_1
✓	0	0	0	1	1
✓	0	1	0	1	1
✓	1	0	1	1	1
✓	1	1	0	1	1

Tabelle 1: Truthtable for Exercise 1 (a).

model	p	q	$\neg p \vee q$	$\neg p \wedge (\neg p \vee q)$	$p \vee \neg q$	φ_2
✓	0	0	1	1	1	1
✗	0	1	1	1	0	0
✗	1	0	0	0	1	0
✗	1	1	1	0	1	0

Tabelle 2: Truthtable for Exercise 1 (b).

model	p	q	r	$p \wedge q$	$p \vee r$	φ_4
✓	0	0	0	0	0	1
✓	0	0	1	0	1	1
✓	0	1	0	0	0	1
✓	0	1	1	0	1	1
✓	1	0	0	0	1	1
✓	1	0	1	0	1	1
✓	1	1	0	1	1	1
✓	1	1	1	1	1	1

Tabelle 3: Truthtable for Exercise 1 (d).

Sample Solution

(a)

$$
\begin{aligned}
& (p \to q) \to (\neg r \wedge q) & \\
\equiv\ & \neg(\neg p \vee q) \vee (\neg r \wedge q) & \text{Definition of '}\to\text{'} \\
\equiv\ & (p \wedge \neg q) \vee (\neg r \wedge q) & \text{De Morgan} \\
\equiv\ & \big((p \wedge \neg q) \vee \neg r\big) \wedge \big((p \wedge \neg q) \vee q\big) & \text{Distribution} \\
\equiv\ & \big((p \vee \neg r) \wedge (\neg q \vee \neg r)\big) \wedge \big((p \vee q) \wedge (\neg q \vee q)\big) & \text{Distribution} \\
\equiv\ & \big((p \vee \neg r) \wedge (\neg q \vee \neg r)\big) \wedge \big((p \vee q) \wedge 1\big) & \text{Complementation} \\
\equiv\ & \big((p \vee \neg r) \wedge (\neg q \vee \neg r)\big) \wedge (p \vee q) & \text{Identity} \\
\equiv\ & (p \vee \neg r) \wedge (\neg q \vee \neg r) \wedge (p \vee q) & \text{Associativity}
\end{aligned}
$$

(b)

$$
\begin{aligned}
& \neg((\neg p \to \neg q) \wedge \neg r) & \\
\equiv\ & \neg((p \vee \neg q) \wedge \neg r) & \text{Definition of '}\to\text{'} \\
\equiv\ & \neg(p \vee \neg q) \vee r & \text{De Morgan} \\
\equiv\ & (\neg p \wedge q) \vee r & \text{De Morgan}
\end{aligned}
$$

Exercise 3: Logical Entailment *(2+2 Points)*

A *knowledge base* KB is a set of formulae over a given set of atoms Σ. An interpretation I of Σ is called a model of KB, if it is a model for *all* formulae in KB. A knowledge base KB *entails* a formula φ (we write $KB \models \varphi$), if *all* models of KB are also models of φ.

Let $KB := \{p \vee q, \neg r \vee p\}$. Show or disprove that KB logically entails the following formulae.

(a) $\varphi_1 := (p \wedge q) \vee \neg(\neg r \vee p)$

(b) $\varphi_2 := (q \leftrightarrow r) \to p$

Sample Solution

(a) KB does not entail φ_1. Consider the interpretation $I : p \mapsto 1, q \mapsto 0, r \mapsto 0$. Interpretation I is a model for KB but not for φ_1.

(b) Table 4 shows that every model of KB is also a model of φ_2, hence $KB \models \varphi_2$.

Exercise 4: Inference Rules and Calculi *(2+2 Points)*

Let $\varphi_1, \ldots, \varphi_n, \psi$ be propositional formulae. An *inference rule*

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}$$

means that if $\varphi_1, \ldots, \varphi_n$ are 'considered true', then ψ is 'considered true' as well ($n = 0$ is the special case of an axiom). A (propositional) *calculus* $\mathbf{C}$ is described by a *set* of inference rules.
Given a formula ψ and knowledge base $KB := \{\varphi_1, \ldots, \varphi_n\}$ (where $\varphi_1, \ldots, \varphi_n$ are formulae) we write $KB \vdash_{\mathbf{C}} \psi$ if ψ can be derived from KB by starting from a subset of KB and repeatedly applying inference rules from the calculus $\mathbf{C}$ to 'generate' new formulae until ψ is obtained.

model of KB	p	q	r	$p \lor q$	$\neg r \lor p$	$q \leftrightarrow r$	φ_2	model of φ_2
✗	0	0	0	0	0	1	0	✗
✗	0	0	1	0	0	0	1	✓
✓	0	1	0	1	1	0	1	✓
✗	0	1	1	1	0	1	0	✗
✓	1	0	0	1	1	1	1	✓
✓	1	0	1	1	1	0	1	✓
✓	1	1	0	1	1	0	1	✓
✓	1	1	1	1	1	1	1	✓

Tabelle 4: Truthtable for Exercise 3 (b).

Consider the following two calculi, defined by their inference rules (φ, ψ, χ are arbitrary formulae).

$$\mathbf{C_1}: \quad \frac{\varphi \to \psi, \psi \to \chi}{\varphi \to \chi}, \frac{\neg\varphi \to \psi}{\neg\psi \to \varphi}, \frac{\varphi \leftrightarrow \psi}{\varphi \to \psi, \psi \to \varphi}$$

$$\mathbf{C_2}: \quad \frac{\varphi, \varphi \to \psi}{\psi}, \frac{\varphi \land \psi}{\varphi, \psi}, \frac{(\varphi \land \psi) \to \chi}{\varphi \to (\psi \to \chi)}$$

Using the respective calculus, show the following derivations (document your steps).

(a) $\{p \leftrightarrow \neg r, \neg q \to r\} \vdash_{\mathbf{C_1}} p \to q$

(b) $\{p \land q, p \to r, (q \land r) \to s\} \vdash_{\mathbf{C_2}} s$

Remark: Inferences of a given calculus are purely syntactical, i.e. rules only apply in their specific form (much like a grammar) and no other logical transformations not given in the calculus are allowed.

Sample Solution

(a) We use $\mathbf{C_1}$ to derive new formulae until we obtain the desired one.

$$\neg q \to r \quad \overset{\text{2nd rule}}{\vdash_{\mathbf{C_1}}} \quad \neg r \to q$$

$$p \leftrightarrow \neg r \quad \overset{\text{3rd rule}}{\vdash_{\mathbf{C_1}}} \quad p \to \neg r, \neg r \to p$$

$$p \to \neg r, \neg r \to q \quad \overset{\text{1st rule}}{\vdash_{\mathbf{C_1}}} \quad p \to q$$

(b) We use $\mathbf{C_2}$ to derive new formulae until we obtain the desired one.

$$p \land q \quad \overset{\text{2nd rule}}{\vdash_{\mathbf{C_2}}} \quad p, q$$

$$p, p \to r \quad \overset{\text{1st rule}}{\vdash_{\mathbf{C_2}}} \quad r$$

$$(q \land r) \to s \quad \overset{\text{3rd rule}}{\vdash_{\mathbf{C_2}}} \quad q \to (r \to s)$$

$$q, q \to (r \to s) \quad \overset{\text{1st rule}}{\vdash_{\mathbf{C_2}}} \quad r \to s$$

$$r, r \to s \quad \overset{\text{1st rule}}{\vdash_{\mathbf{C_2}}} \quad s$$

Exercise 5: Resolution Calculus *(1+1+3 Points)*

Due to the *Contradiction Theorem* (cf. lecture) for every knowledge base KB and formula φ it holds

$$KB \models \varphi \quad \Longleftrightarrow \quad KB \cup \{\neg\varphi\} \models \bot.$$

Remark: $\bot$ is a formula that is unsatisfiable.

In order to show that KB entails φ, we show that $KB \cup \{\neg\varphi\}$ entails a contradiction. A calculus **C** is called *refutation-complete* if for every knowledge base KB

$$KB \models \bot \quad \Longrightarrow \quad KB \vdash_{\mathbf{C}} \bot.$$

Hence, given a refutation-complete calculus **C** it suffices to show $KB \cup \{\neg\varphi\} \vdash_{\mathbf{C}} \bot$ to prove $KB \models \varphi$.

The *Resolution Calculus* **R** is a formal way to do a prove by contradiction. It is correct and refutation-complete[1] for knowledge bases that are given in *Conjunctive Normal Form* (CNF). A knowledge base KB is in CNF if it is of the form $KB = \{C_1, \ldots, C_n\}$ where its clauses $C_i = \{L_{i,1}, \ldots, L_{i,m_i}\}$ each consist of m_i literals $L_{i,j}$.
Remark: KB represents the formula $C_1 \wedge \ldots \wedge C_n$ with $C_i = L_{i,1} \vee \ldots \vee L_{i,m_i}$.

The Resolution Calculus has only one inference rule, the *resolution rule*:

$$\mathbf{R}: \quad \frac{C_1 \cup \{L\}, C_2 \cup \{\neg L\}}{C_1 \cup C_2}.$$

Remark: L is a literal and $C_1 \cup \{L\}, C_2 \cup \{\neg L\}$ are clauses in KB (C_1, C_2 may be empty). To show $KB \vdash_{\mathbf{R}} \bot$, you need to apply the resolution rule, until you obtain two conflicting one-literal clauses L and $\neg L$. These entail the empty clause (defined as $\square$), i.e. a contradiction ($\{L\}, \{\neg L\} \vdash_{\mathbf{R}} \bot$).

(a) We want to show $\{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow u\} \models u$. First convert this problem instance into a form that can be solved via resolution as described above. Document your steps.

(b) Now, use resolution to show $\{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow u\} \models u$.

(c) Consider the sentence "Heads, I win". "Tails, you lose". Design a propositional KB that represents these sentences (create the propositions and rules required). Then use propositional resolution to prove that **I always win**.

Sample Solution

(a) We transform $\{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow u\} \models u$ into the form $KB \models \bot$ where KB is in CNF. The given entailment is equivalent to $\{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow u, \neg u\} \models \bot$ using the Contradiction Theorem, which we described above. Now we transform the knowledge base into CNF using DeMorgan's rule and distribution among others.

$$\begin{aligned} & \{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow u, \neg u\} \\ \equiv\ & \{p, q, \neg p \vee r, \neg(q \wedge r) \vee u, \neg u\} \\ \equiv\ & \{p, q, \neg p \vee r, \neg q \vee \neg r \vee u, \neg u\} \\ \equiv\ & \{\{p\}, \{q\}, \{\neg p, r\}, \{\neg q, \neg r, u\}, \{\neg u\}\} \end{aligned}$$

(b) Now we can use the Resolution calculus **R** to derive a contradiction (the empty clause $\square$).

$$\begin{aligned} \{\neg p, r\}, \{p\} &\vdash_{\mathbf{R}} \{r\} \\ \{\neg q, \neg r, u\}, \{r\} &\vdash_{\mathbf{R}} \{\neg q, u\} \\ \{\neg q, u\}, \{\neg u\} &\vdash_{\mathbf{R}} \{\neg q\} \\ \{\neg q\}, \{q\} &\vdash_{\mathbf{R}} \square \end{aligned}$$

We have a *contradiction.* Thus, the above entailment is true.

[1] Complete calculi are impractical, since they have too many inference rules. More inference rules make automated proving with a computer significantly more complex. The Resolution Calculus is an appropriate technique to avoid this additional complexity, since it has only one inference rule.

(c) 1) Define the atomic formulae from text above: H : heads T : tails I : I win Y : You win.

2) Use these to state the rules: $H \to I$ and $T \to \neg Y$.

3) We now must specify implicit rules. The formulas above do not yet know that heads and tails are mutually exclusive: $H \otimes T$ and $I \otimes Y$ ($A \otimes B := (A \vee B) \wedge (\neg A \vee \neg B)$ is the XOR operator).

4) Convert to CNF:

$$\begin{aligned} & H \to I \text{ and } T \to \neg Y \text{ and } H \otimes T \text{ and } I \otimes Y \\ \equiv\; & \neg(H \vee I) \;\wedge\; (\neg T \vee \neg Y) \;\wedge\; (H \vee T) \wedge (\neg H \vee \neg T) \;\wedge\; (I \vee Y) \wedge (\neg I \vee \neg Y) \\ \equiv\; & \{\{\neg H, I\}, \{\neg T, \neg Y\}, \{H, T\}, \{\neg H, \neg T\},\; \{I, Y\},\; \{\neg I, \neg Y\}\} \end{aligned}$$

5) We want to prove I, hence we add the literal $\{\neg I\}$ to the knowledge base:

$$\{\{\neg H, I\}, \{\neg T, \neg Y\}, \{H, T\}, \{\neg H, \neg T\},\; \{I, Y\},\; \{\neg I, \neg Y\}, \{\neg I\}\}.$$

Now we start resolving clauses:

$$\begin{aligned} \{\neg T, \neg Y\}, \{H, T\} &\quad \vdash_{\mathbf{R}} \quad \{H, \neg Y\} \\ \{H, \neg Y\}, \{\neg H, I\} &\quad \vdash_{\mathbf{R}} \quad \{I, \neg Y\} \\ \{\neg I\}, \{I, \neg Y\} &\quad \vdash_{\mathbf{R}} \quad \{\neg Y\} \\ \{\neg Y\}, \{I, Y\} &\quad \vdash_{\mathbf{R}} \quad \{I\} \\ \{I\}, \{\neg I\} &\quad \vdash_{\mathbf{R}} \quad \Box \end{aligned}$$

Consequently, we have a *contradiction*. Thus, I is true.