



# Theoretical Computer Science - Bridging Course

## Sample Solution Exercise Sheet 11

Due: Tuesday, 23rd of January 2025, 12:00 pm

### Exercise 1: Construct Formulae

(1+1+1 Points)

Let  $\mathcal{S} = \langle \{x, y, z\}, \emptyset, \emptyset, \{R\} \rangle$  be a signature. Translate the following sentences of first order formula over  $\mathcal{S}$  into idiomatic English. Use  $R(x, y)$  as statement 'x is a part of y'.

- (a)  $\exists x \forall y R(x, y)$ .
- (b)  $\exists y \forall x R(x, y)$ .
- (c)  $\forall x \forall y \exists z (R(x, z) \wedge R(y, z))$

### Sample Solution

Note that idiomatic English uses, contains, or denotes expressions that are natural to a native speaker. It does not contain variables. it might be said: A native speaker will then read the formulas as

- (a) Something is a part of everything.
- (b) Something has everything as a part.
- (c) For any two things, there is something of which they are both a part.

### Exercise 2: FOL: Is it a model?

(2+3+3 Points)

Consider the following **first order** formulae

$$\begin{aligned}\varphi_1 &:= \forall x R(x, x) \\ \varphi_2 &:= \forall x \forall y R(x, y) \rightarrow (\exists z R(x, z) \wedge R(z, y)) \\ \varphi_3 &:= \exists x \exists y (\neg R(x, y) \wedge \neg R(y, x))\end{aligned}$$

over signature  $\mathcal{S}$  where  $x, y, z$  are variable symbols and  $R$  is a binary predicate. Give an interpretation

- (a)  $I_1$  which is a **model** of  $\varphi_1 \wedge \varphi_2$ .
- (b)  $I_2$  which is **no model** of  $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ .
- (c)  $I_3$  which is a **model** of  $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ .

## Sample Solution

- (a) Pick  $I_1 := \langle \mathbb{R}, \cdot^{I_1} \rangle$  where  $R^{I_1}(x, y) : \iff x \leq_{\mathbb{R}} y$ .

This is a model because ' $\leq_{\mathbb{R}}$ ' is *reflexive*, therefore fulfills  $\varphi_1$ . Moreover for every  $x, y \in \mathbb{R}$  with  $x \leq_{\mathbb{R}} y$  we can choose  $z := x$ , which fulfills  $x \leq_{\mathbb{R}} z \wedge z \leq_{\mathbb{R}} y$ . Thus  $\varphi_2$  is also satisfied.

- (b) Let  $U = \{1, 2, 3, 4, 5\}$  and  $P(U)$  be its power set. Pick  $I_2 := \langle P(U), \cdot^{I_2} \rangle$  where  $R^{I_2}(x, y) : \iff x \subset y$ . This is not a model since it doesn't satisfy  $\varphi_1$ , indeed no set is proper subset of itself.

- (c) Take two disjoint copies of  $\mathbb{R}$  and the standard  $\leq_{\mathbb{R}}$  relation on each of them; if  $x$  and  $y$  are from different copies they are not related in  $\mathbb{R}$ . Formally let

$$I_3 := \langle \{(a, 1) \mid a \in \mathbb{R}\} \dot{\cup} \{(a, 2) \mid a \in \mathbb{R}\}, \cdot^{I_3} \rangle$$

where  $R^{I_3}((a, g), (b, h)) \iff (g = h \text{ and } a \leq_{\mathbb{R}} b)$ .

This is a model because  $\leq_{\mathbb{R}}$  is *reflexive*, therefore  $I_3$  fulfills  $\varphi_1$ . Furthermore for every two  $x = (a, g)$  and  $y = (b, h)$  with  $R^{I_3}((a, g), (b, h))$ , i.e.,  $g = h$ , we can choose  $z := (a, g)$  which fulfills  $R^{I_3}((a, g), (a, g)) \wedge R^{I_3}((a, g), (b, h))$ . Thus  $\varphi_2$  is also satisfied.  $\varphi_3$  is also satisfied, e.g.,  $(5, 1)$  and  $(7, 2)$  are incomparable, i.e., we have neither  $R^{I_3}((5, 1), (7, 2))$  nor  $R^{I_3}((7, 2), (5, 1))$ .

## Exercise 3: FOL: Entailment

(3+3+3 Points)

Let  $\varphi, \psi$  be first order formulae over signature  $\mathcal{S}$ . Similar to propositional logic, in predicate logic we write  $\varphi \models \psi$  if every model of  $\varphi$  is also a model for  $\psi$ . We write  $\varphi \equiv \psi$  if both  $\varphi \models \psi$  and  $\psi \models \varphi$ . A *knowledge base*  $KB$  is a set of formulae. A model of  $KB$  is model for all formulae in  $KB$ . We write  $KB \models \varphi$  if all models of  $KB$  are models of  $\varphi$ . Show or disprove the following entailments.

- (a)  $(\exists x R(x)) \wedge (\exists x P(x)) \wedge (\exists x T(x)) \models \exists x (R(x) \wedge P(x) \wedge T(x))$ .
- (b)  $(\forall x \forall y f(x, y) \doteq f(y, x)) \wedge (\forall x f(x, \mathbf{c}) \doteq x) \models \forall x f(\mathbf{c}, x) \doteq x$ .
- (c)  $(\forall x R(x, x)) \wedge (\forall x \forall y R(x, y) \wedge R(y, x) \rightarrow x \doteq y) \wedge (\forall x \forall y \forall z R(x, y) \wedge R(y, z) \rightarrow R(x, z))$   
 $\models \forall x \forall y R(x, y) \vee R(y, x)$ .

*Hint: Consider order relations. E.g.,  $a \leq b$  (a less-equal b) and  $a \mid b$  (a divides b).*

## Sample Solution

- (a) The stated entailment is false (it holds in the other direction though). In order to disprove it, we give a model for the left side which is not a model for the right side.

Let  $I = \langle \{a, b\}, \cdot^I \rangle$  with  $R^I = \{a\}$ ,  $P^I = \{b\}$ , and  $T^I = \{a\}$ . This makes the left side true since there exists an element  $x = a$  that makes  $R(x)$  and  $T(x)$  true and an element  $x = b$  that makes  $P(x)$  true (Note the brackets around the three  $\exists$  quantifiers which mean that the three elements need not necessarily be the same).

However  $R(a) \wedge P(a) \wedge T(a) = T \wedge F \wedge T = F$  and  $R(b) \wedge P(b) \wedge T(b) = F \wedge T \wedge F = F$  thus the right side is false (there exists no element which makes the three relations' symbols  $R, P, T$  true, since we tested all that are in the domain).

- (b) The stated entailment holds. We prove this by picking an arbitrary *model* (!)  $I = \langle \mathcal{D}, \cdot^I \rangle$  of the left-hand formula. We show that  $I$  is a model for the right-hand formula, too. For that purpose let  $x$  be an arbitrary element from  $\mathcal{D}$ .

Since  $I$  is a model for the left side we already know  $f(x, \mathbf{c}^I) \doteq x$ . The first condition in the left formula encodes the commutative property. Since  $\mathbf{c}^I$  is also an element from the domain  $\mathcal{D}$  we know  $f(x, \mathbf{c}^I) = f(\mathbf{c}^I, x)$  and thus  $f(\mathbf{c}^I, x) \doteq x$ . Since  $x$  was arbitrary we have  $\forall x f(\mathbf{c}^I, x) \doteq x$ .

- (c) The formula on the left side encodes the properties of an *order relation*. The formula on the right side encodes the property of *totality* of an order, which means that every element is related to (read: can be compared with) every other element. However, in general an order relation does not need to be total (which is called a *partial order*).

The hint proposes two order relations, one of which is total over the domain of integers  $\mathbb{Z}^*$  (either ' $x \leq y$ ' or ' $x \leq y$ ' or both) whereas the other is not (it may happen that neither  $x|y$  nor  $x|y$ ). Thus the logical entailment is false since with  $\mathbb{Z}^*$  and the 'divides'-relation we have a model of the left-hand formula which is no model of the right-hand one (it is not total).

We formalize this as follows. Let  $I = \langle \mathbb{Z}^*, \cdot^I \rangle$  with  $R^I := \{(x, y) \in \mathbb{Z}^* \mid x \text{ divides } y\}$ . Obviously we have the reflexive property since  $x \in \mathbb{Z}^*$  divides itself. If  $x \in \mathbb{Z}^*$  divides  $y \in \mathbb{Z}^*$  and  $y \in \mathbb{Z}^*$  divides  $z \in \mathbb{Z}^*$  then  $x$  also divides  $z$  which gives us transitivity. Finally, if  $x$  divides  $y$  and vice versa then  $y$  is multiple of  $x$  and vice versa which means that the multiplicand must in both cases be 1, thus both  $x$  and  $y$  are equal which gives us the antisymmetry property.

This means that  $I$  is a model of the left-hand formula. Now consider the two primes  $x = 2$  and  $y = 3$ . By definition of prime numbers neither of the two can divide the other. Thus  $\forall x \forall y R(x, y) \vee R(y, x)$  is false. Therefore  $I$  can be no model of the right-hand formula.