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Algorithm Theory
Exercise Sheet 9

Due: Friday, 19th of December 2025, 10:00 am

Exercise 1: Fully Saturated Edges? *(14 Points)*

Let $G = (V, E)$ be an $s - t$ flow network with integer capacity's $c_e > 0$ on each edge $e \in E$. We say that edge $e \in E$ is *fully saturated* if the flow uses its full capacity in every maximum $s - t$ flow in G.

(a) Assume we are given a maximum flow f^* of G, describe an algorithm with time complexity $\mathcal{O}(|E|)$ that computes a new maximum flow of G when the capacity of a specific edge $\{u, v\} \in E$ with $c_e \geq 1$ decreases by one unit. *(3 Points)*

(b) Prove that any minimum cut of G exclusively contains fully saturated edges. *(2 Points)*

(c) Prove that an edge $e \in E$ is fully saturated *if and only if* decreasing the capacity of e by 1 decreases the maximum flow value in G. *(5 Points)*
Remark: Notice that in general even if the maximum flow value and the capacities are integers, the flow values per edges don't necessarily need to be integers. However, the values of (min) cuts are always integers. It may make sense to consider cuts for the backward direction and use the result of (b).

(d) Devise an algorithm that computes the number of fully saturated edges in the flow network G and analyze its running time in dependency of E and the value of the maximum flow $|f^*|$. *(4 Points)*
Remark: Use the statement of (c) and the algorithm from (a) as a black box.

Exercise 2: Work Schedule *(6 Points)*

Assume you want to design a work schedule for a hospital for the next n days. The hospital employs k doctors. On day $1 \leq i \leq n$, **exactly** p_i doctors need to be present in the hospital as less doctors can not provide an optimal health service and more doctors would be too expensive. Each doctor $1 \leq j \leq k$ provides a set $L_j \subseteq \{1, \ldots, n\}$ of days on which they are willing to work.

(a) Describe a polynomial-time algorithm that either: *(3 Points)*

- Returns a list $L'_j \subseteq L_j$ of working days for each doctor j such that on day i, exactly p_i doctors are present; or
- Reports that there is no such set of lists that fulfills the given constraints.

(b) The hospital finds that the doctors tend to submit lists that are too restrictive, and consequently, it often happens that there is no feasible working schedule. Thus, the hospital relaxes the requirements in the following way:

Each doctor j can be forced to work on up to 5 days which are *not* in their list L_j.

Give a polynomial-time algorithm to solve this problem. The algorithm should either: *(3 Points)*

- Return a list L'_j of working days for each doctor j with $|L'_j \setminus L_j| \leq 5$ such that on day i, exactly p_i doctors are present; or
- Report that there is no such set of lists that fulfills the given constraints.