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Algorithm Theory
Exercise Sheet 10

Due: Friday, 9th of January, 2026, 10:00 am

Exercise 1: Perfect Matchings in regular Graphs *(10 Points)*

We call an undirected graph d-regular if each node has exactly d edges. Let $G = (A \cup B, E)$ be a bipartite d-regular graph.
Show that E is the union of d perfect matchings, i.e., show that $E = E_1 \cup \cdots \cup E_d$, where $E_1, \ldots, E_d$ are perfect matchings in G.
Hint: First show that the preconditions to apply Hall's Theorem hold and then think of a way to use it here.

Exercise 2: Cover a bipartite Graph *(10 Points)*

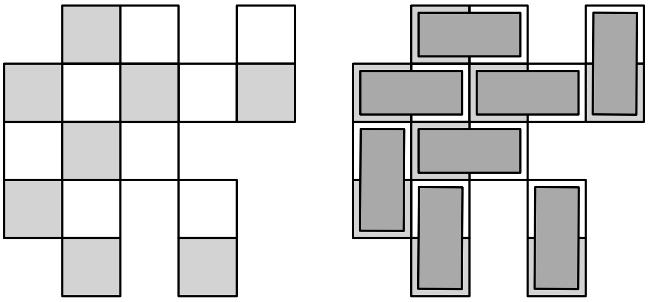
Let $G = (A \cup B, E)$ be a bipartite graph and M a matching of E. We say M **covers** A if each node $u \in A$ is adjacent to one edge $e \in M$.

(a) Prove the following statement:
In a bipartite graph $G = (A \cup B, E)$ there exists a matching M that covers A if and only if $|N(S)| \geq |S|$ for all sets $S \subseteq A$. *(7 Points)*
Hint: Note that for $|A| = |B|$ the theorem is equivalent to Halls Theorem. If $|A| < |B|$ you can try to adjust the graph such that one can still apply Halls Theorem.

(b) For a given integer $d \geq 1$, let $G_d = (A \cup B, E)$ be a bipartite graph where for all $v \in A : deg(v) \geq d$ and for all $u \in B : deg(u) = d$. Show that for each such graph G_d there exists a matching M that covers A. *(3 Points)*
Hint: Use the statement of (a).

Exercise 3: Checkerboard *(4 Bonus Points)*

Consider an incomplete $n \times n$ checkerboard, i.e., where some tiles are cut out. The incomplete checkerboard is given by an $n \times n$ array C with $C[i][j] = 0$ if the tile at position $(i, j) \in \{0, \ldots, n-1\}^2$ has been cut out, else $C[i][j] = 1$. We want to answer whether we can place domino pieces, each of which covers *exactly* two adjacent tiles on the checkerboard, such that all tiles are covered (for instance in the example below the answer is yes). More precisely, we want to cover every tile (i, j) with $C[i][j] = 1$ with some domino piece, such that domino pieces *do not overlap* and *only cover existing tiles* ($C[i][j] = 1$ with $i, j \in \{0, \ldots, n-1\}^2$). Give an efficient algorithm that answers this question and argue why it is correct. *(7 Points)*



Exercise 4: Chess tournament *(6 Bonus Points)*

Assume that there are n chess players $1, \ldots, n$ that you need to pair up for playing against each other in a chess tournament.
There are some players who must play their next game with the white pieces and there are some players who must play their next game with the black pieces. There are also players for whom it does not matter if they play with the white or the black pieces.
In addition, each player i has a rating value r_i, which is a positive integer.
Each chess game in the tournament must be played between exactly two players: one playing with the white pieces and the other with the black pieces. Further, each player should play in at most one game. Additionally, to ensure balanced games, the absolute difference in rating between the two players in a game must be smaller than 100.

(a) Describe a polynomial-time algorithm to determine a largest possible set of chess games that can be arranged with the available n players. You can use algorithms from the lecture as a black box. *(3 Points)*

(b) Assume that we make the (strange) requirement that the absolute difference in rating between the two players of a game must be an odd number < 100? Argue why the problem of determining a maximum set of possible chess games now becomes easier! *(3 Points)*