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Algorithm Theory
Exercise Sheet 12

Due: Friday, 30th of January, 2026, 10:00 am

Exercise 1: k-th smallest element *(10 Points)*

Given a set S of n pairwise distinct numbers and a number $k \in \{1, \ldots, n\}$, we want to define a function that returns the k^{th} smallest element of S.
Consider the following randomized divide and conquer algorithm.

Algorithm 1 `select`(S, k)

Choose a pivot $x \in S$ uniformly at random
$S^- = \emptyset$
$S^+ = \emptyset$
for all $y \in S \setminus \{x\}$ **do**
    if $y < x$ **then**
        $S^- = S^- \cup \{y\}$
    else
        $S^+ = S^+ \cup \{y\}$
if $|S^-| = k - 1$ **then**
    return x
else
    if $|S^-| \geq k$ **then**
        return `select`(S^-, k)
    else
        if $|S^-| < k - 1$ **then**
            return `select`$(S^+, k - |S^-| - 1)$

(a) Shortly explain why `select`(S, k) is correct, i.e., returns the k^{th} smallest element of S, and explain the worst-case runtime. *(3 Points)*

(b) What is the probability that both $|S^+| \leq \frac{3}{4}|S|$ and $|S^-| \leq \frac{3}{4}|S|$ (after one iteration)? Explain your answer. For simplicity, you may assume that $|S|$ is a multiple of 4. *(1 Point)*

(c) Give an upper bound on the expected runtime of `select`(S, k). Explain your answer. *(3 Points)*
Hint: Use (b).

(d) Show that `select`(S, k) terminates in time $O(n \log n)$ with probability at least $1 - \frac{1}{n}$. *(3 Points)*

Hint: Use the following Chernoff's Bound: If $X_1, \ldots, X_n$ is a sequence of independent 0-1 random variables, $X = \sum X_i$ and $\mu = E[X]$, then for any $0 < \delta < 1$ we have

$$\Pr(X \leq (1 - \delta)\mu) \leq e^{-\frac{\delta^2}{2}\mu}.$$

Exercise 2: Randomized Coloring *(10 Points)*

Let $G = (V, E)$ be a simple, undirected graph with maximum degree Δ. A (node) coloring of the graph is an assignment of colors to the nodes in a way that no two adjacent nodes are assigned with the same color. More formal: A coloring is a mapping $\phi : V \to C$ of nodes in V to some color space C s.t. $\phi(u) \neq \phi(v)$ if $\{u, v\} \in E$.
Consider Algorithm 2 to assign colors from the colors pace $\{1, 2, \ldots, \Delta + 1\}$ to the nodes. Let L_v be the lists of **available** colors of v, that initially is set to the color space.

Algorithm 2 Randomized Coloring

Ensure: ϕ is a proper $\Delta + 1$ coloring
Let $L_v := \{1, 2, \ldots, \Delta + 1\}$
for each uncolored node $v \in V$ in parallel **do**
  v becomes active with probability $p = \frac{1}{2}$
  if v is active **then**
    Let v choose a color $x_v \in L_v$ uniformly at random
    if no neighbor u picked x_v as well **then**
      $\phi(v) := x_v$ ▷ v is colored now!
  if v is still uncolored **then**
    delete $\phi(u)$ from L_v for all colored neighbors u. ▷ Update L_v

Note that in every iteration, $|L_v|$ is larger than the number of uncolored neighbors of v.

(a) Show that a node v that is still uncolored will be colored in the next iteration with probability at least $1/4$. *(5 Points)*
Hint: Assume v is active and has k uncolored neighbors. What is the probability that v gets colored?

(b) After how many iterations is a node $v \in V$ colored in expectation? *(2 Points)*

(c) Show that Algorithm 2 terminates in $O(\log n)$ iterations **with high probability**.
That is for a given constant $c > 0$, all nodes are colored within $O(\log n)$ iterations with probability at least $1 - \frac{1}{n^c}$. *(3 Points)*
Hint: Use the result of (a) for tasks (b) and (c) even if you didn't manage to come up with a solution.