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Algorithm Theory
Sample Solution Exercise Sheet 10

Due: Friday, 9th of January, 2026, 10:00 am

Exercise 1: Perfect Matchings in regular Graphs *(10 Points)*

We call an undirected graph d-regular if each node has exactly d edges. Let $G = (A \cup B, E)$ be a bipartite d-regular graph.
Show that E is the union of d perfect matchings, i.e., show that there exists a paritioning of the edges $E = E_1 \cup \cdots \cup E_d$, where $E_1, \ \ldots, E_d$ are perfect matchings in G.
Hint: First show that the preconditions to apply Hall's Theorem hold and then think of a way to use it here.

Sample Solution

We first want to apply Hall's Theorem to show that there exist perfect matchings. To apply Hall's Theorem we need to make sure we meet two criteria (i) $|A| = |B|$ and (ii) $\forall A' \subseteq A : |N(A')| \geq |A'|$. In our case since we assume only d-regularity it suffices to show (ii) since it automatically implies (i) through symmetry (more detailed, if we assume $|A| > |B|$ then some B node needs to have degree strictly larger than d, contradiction). As G is d-regular and bipartite, each node in A has to have d neighbors in B and vice versa. Let A' be an arbitrary subset of A and let $E_{A'}$ be the edges between A' and its neighbors. By the regularity we know $|E_{A'}| = d \cdot |A'|$. The nodes in $N(A')$ also have at most d edges each, but some of them my connect so some node in $A \setminus A'$. The following inequality takes that into account: $d \cdot |N(A')| \geq E_{A'}$. We can conclude that $d \cdot |A'| = |E_{A'}| \leq d \cdot |N(A')|$ and so $|A'| \leq |N(A')$.
Thus, we have shown that the preconditions for Hall's Theorem hold. We will now use it to show that there exists some perfect matching E_d in G, where the subscript d denotes the fact that we have obtained it in a $d-$regular graph (this notation is useful for recursion purposes). By the definition of a matching each node is only incident to at most one edge from E_d and since it is a perfect matching also every node has an edge that is in E_d. We now simply delete this matching from G, as we just discussed, every node has exactly one adjacent edge in E_d, therefore the degree of each node gets reduced by 1. Since the graph was d-regular before, the updated one is a $(d-1)$-regular graph G'. Note tha G' is not nessesarily a connected graph anymore, but all it's connected components are still regular bipartite graphs. Hence, we can just iterate the above procedure, i.e., compute a perfect matching E_{d-1} on G', delte these edges from G' and recurse, until we have deleted all of the edges. We can repeat this exactly d times (as G has $m = d \cdot \frac{n}{2}$ edges and in every iteration we delete $n/2$ many edges) and hence also get that $E_d \cup E_{d-1} \cup \cdots \cup E_1 = E$ as claimed.

Exercise 2: Cover a bipartite Graph *(10 Points)*

Let $G = (A \cup B, E)$ be a bipartite graph and M a matching of E. We say M **covers** A if each node $u \in A$ is adjacent to one edge $e \in M$.

(a) Prove the following statement:
In a bipartite graph $G = (A \cup B, E)$ there exists a matching M that covers A if and only if $|N(S)| \geq |S|$ for all sets $S \subseteq A$. *(7 Points)*
Hint: Note that for $|A| = |B|$ the theorem is equivalent to Halls Theorem. If $|A| < |B|$ you can try to adjust the graph such that one can still apply Halls Theorem.

(b) For a given integer $d \geq 1$, let $G_d = (A \cup B, E)$ be a bipartite graph where for all $v \in A : deg(v) \geq d$ and for all $u \in B : deg(u) = d$. Show that for each such graph G_d there exists a matching M that covers A. *(3 Points)*
Hint: Use the statement of (a).

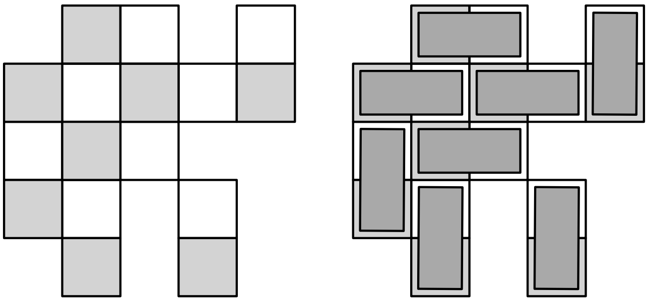
Sample Solution

(a) Direction $\Rightarrow$:
Here we have the guarantee that such a matching M exists. Thus, for each set $S \subseteq A$, each node $v \in S$ has matched edge going to some node $u \in B$ where no other matched edge from S can end as well (as by the definition of a matching, there is only one matched edge per node). Thus, if we just consider the neighbors of S induced by the matched edges we get a set of size exactly $|S|$. So the overall number of neighbors $|N(S)|$ is at least $|S|$.
Direction $\Leftarrow$:
If $|A| = |B|$ the condition for Hall's Theorem is fulfilled and hence there exists a perfect matching between A and B (that obviously covers A and B) and thus it's true. If $|A| < |B|$ we construct a new graph $G' = (A' \cup B, E)$ that contains all nodes and edges from G but adds new nodes. Indeed, we add nodes to A such that both sides again have the same size i.e. ($|A'| = |B|$) and we connect all those new nodes in A' with all nodes in B. It is not hard to see that we now have $|A'| = |B|$ and that for all $S \subset A'$ also $|S| \leq |N(S)|$ (since we know by our assumption that this property holds if $S \subseteq A$, and if S contains one of the new nodes we have $|N(S)| = |B|$ which also makes the statement true). Thus, we can apply Hall's Theorem on G' and get a perfect matching M' that covers A'. If we now consider only the matched edges of the original nodes in A, this matching is a valid matching and is incident to each node in A. Thus, the statement is true.

(b) First note that $|A| \leq |B|$ by the degree constraints. We will now show that for all $S \subseteq A$ we have $|N(S)| \geq |S|$. Note that each node in S has degree $\geq d$. So we know that the number of edges between S and $|N(S)|$ are $m_S \geq d \cdot |S|$. In the worst case each node of $N(S)$ is an endpoint of exactly d edges from nodes in S. Thus, $m_S \leq d \cdot |N(S)|$. Putting both inequalities together and dividing by d we get $|S| \leq |N(S)|$. We can now apply (a) to end the proof.

Exercise 3: Checkerboard *(4 Bonus Points)*

Consider an incomplete $n \times n$ checkerboard, i.e., where some tiles are cut out. The incomplete checkerboard is given by an $n \times n$ array C with $C[i][j] = 0$ if the tile at position $(i, j) \in \{0, \ldots, n-1\}^2$ has been cut out, else $C[i][j] = 1$. We want to answer whether we can place domino pieces, each of which covers *exactly* two adjacent tiles on the checkerboard, such that all tiles are covered (for instance in the example below the answer is yes). More precisely, we want to cover every tile (i, j) with $C[i][j] = 1$ with some domino piece, such that domino pieces *do not overlap* and *only cover existing tiles* ($C[i][j] = 1$ with $i, j \in \{0, \ldots, n-1\}^2$). Give an efficient algorithm that answers this question and argue why it is correct. *(7 Points)*



Sample Solution

The problem corresponds to finding a perfect matching in a bipartite graph, where the nodes of the bipartition correspond to the white and black tiles of the checkerboard respectively with an edge between horizontally or vertically adjacent tiles. The incomplete checkerboard can be covered if and only if there is a perfect matching. We know from the lecture that we can compute a maximum matching in bipartite graphs efficiently, in number of edges times number of nodes. Here, the number of nodes is at most $O(n^2)$ and since each node has a degree of at most 4 to the other side, also the number of edges is at most $O(n^2)$. Note that this means that the overall runtime is bounded by $O(n^4)$.

Exercise 4: Chess tournament *(6 Bonus Points)*

Assume that there are n chess players $1, \ldots, n$ that you need to pair up for playing against each other in a chess tournament.
There are some players who must play their next game with the white pieces and there are some players who must play their next game with the black pieces. There are also players for whom it does not matter if they play with the white or the black pieces.
In addition, each player i has a rating value r_i, which is a positive integer.
Each chess game in the tournament must be played between exactly two players: one playing with the white pieces and the other with the black pieces. Further, each player should play in at most one game. Additionally, to ensure balanced games, the absolute difference in rating between the two players in a game must be smaller than 100.

(a) Describe a polynomial-time algorithm to determine a largest possible set of chess games that can be arranged with the available n players. You can use algorithms from the lecture as a black box. *(3 Points)*

(b) Assume that we make the (strange) requirement that the absolute difference in rating between the two players of a game must be an odd number < 100? Argue why the problem of determining a maximum set of possible chess games now becomes easier! *(3 Points)*

Sample Solution

(a) We construct a graph $G = (V, E)$, where we add a node for every player. We add an edge between two nodes if the rating difference is smaller than 100 and if either one of them is a 'white' and the other a 'black' player or if at least one of them can play with both colors. A matching on G solves the problem. To compute it, we can use the Edmond's Blossom Algorithm as this runs in time $O(mn^2)$.

(b) The graph G, as constructed above plus the new requirement, always constructs a bipartite graph. On the 'left' side are the players with even ratings and on the 'right' there are players with odd ratings. There can not exist an edge on the left side as the difference of their ratings would be even and for the same reason there can not exist an edge between nodes on the right side. This problem can be solved in time $O(m \cdot n)$.