University of Freiburg
Dept. of Computer Science
Prof. Dr. F. Kuhn
Z. Parsaeian, G. Schmid

Distributed Graph Algorithms
Exercise Sheet 5

Exercise 1: Exact Maximum Matching in Bipartite Graphs *(20 Points)*

In this exercise, we want to design a distributed algorithm in the CONGEST model for computing an **exact maximum (unweighted) matching** that runs in $O(s^\star \log s^\star)$ rounds, where $s^\star$ is the size of a maximum matching. We first recall the notion of an *augmenting path.*

Augmenting Paths: For a given matching M of a graph $G = (V, E)$, an augmenting path $P = u_0, u_1, \dots, u_{2t+1}$ is an odd-length path that starts and ends in an unmatched node and that alternates between edges not in M and edges in M. Given an augmenting path P, one can obtain a matching M' of size $|M'| = |M| + 1$ by flipping the role of each edge along the path (i.e., edges not in M are added to the matching and edges in M are removed from the matching). A matching M is a maximum matching if and only if there is no augmenting path w.r.t. M.

We will develop the maximum matching algorithm in several steps. If you are not able to prove the claim of one of the steps, you can just assume it is true and use it in the susequent steps.

(a) Let $G = (V, E)$ be a graph with maximum matching size $s^\star$. Prove that if a given matching M has size $|M| = s^\star - k$ for some integer $k \geq 1$ then there exists an augmenting path of length at most $\frac{2s^\star}{k}$.

Hint: Consider the symmetric difference between M and some maximum matching of size s^*.

(b) Given a bipartite graph $G = (V, E)$ with diameter D, give a CONGEST algorithm to compute the bipartition of the graph in $O(D)$ rounds (i.e., partition the nodes V in to V_L and V_R such that all edges are between V_L and V_R).

(c) Given a graph $G = (V, E)$ with maximum matching size $s^\star$, give a $O(D + s^\star)$-rounds CONGEST algorithm to determine an upper bound $\hat{s}$ on $s^\star$ such that $s^\star \leq \hat{s} \leq 2s^\star$.

(d) Given a graph $G = (V, E)$ with maximum matching size $s^\star$ and diameter D, show that it always holds that $s^\star = \Omega(D)$. As a consequence, all the steps up to here require at most $O(s^\star)$ rounds.

(e) Assume that that we are given a bipartite graph $G = (V_L \cup V_R, E)$, a matching M of G, and an integer parameter L. You can further assume that all nodes of G know if they are in V_L or in V_R. Give an $O(L)$-round CONGEST algorithm that computes a new matching M' with the following properties. If G has an augmenting path of length at most L w.r.t. M, then $|M'| > |M|$. Otherwise, the matching has to remain the same, i.e., $M' = M$.

(f) We now have all the necessary tools to build our algorithm. Starting from an empty matching $M = \emptyset$, the algorithm operates in $\log_2 \hat{s}$ phases $i = 1, 2, \dots, \lceil \log_2 \hat{s} \rceil$. Phase i consists of $\lceil \hat{s}/2^i \rceil$ interations. In each of those iterations, we run the algorithm of (e) with parameter $L_i = 2^{i+1}$.

Show that this algorithm computes a maximum matching in time $O(s^\star \log s^\star)$.

Hint: Show that at the end of phase i, the matching has size at least $s^\star - s^\star/2^i$.