University of Freiburg
Dept. of Computer Science
Prof. Dr. F. Kuhn
Z. Parsaeian, G. Schmid

Distributed Graph Algorithms
Exercise Sheet 6

LCLs in paths and Cycles

In this exercise sheet, we want to talk about what possible complexities **L**ocally **C**heckable **L**abelings (LCLs) can have, when we restrict the networks to only path and cycle topologies.

1 Locally Checkable Labelings

An LCL is a tuple (Σ, r, C) comprised of the following

- A set of outputlabels Σ (e.g. for 3-coloring $\Sigma = \{R, G, B\} \equiv \{1, 2, 3\}$)
- An integer $r \in \mathbb{N}$ called the radius.
- A set C of Σ labeled r-hop balls.

That is, each element of the set C is just a center node, with an arbitrary r-hop neighborhood, where each node in that neighborhood is assigned an output from Σ.
The set C is the set of *allowed* neighborhoods and we call each such neighborhood a configuration. In a solution to our problem, the output-labeled r-hop neighborhood $\mathcal{N}_r(v)$ of every node v in our graph, must be isomorphic to (so must look like) one of the neighborhoods in C.
Two such set of configuraitons are highlighted in Figures 1 and 2 for the MIS and 3 coloring problems in cycles. Note that for both of these problems the radius is equal to one ($r = 1$).

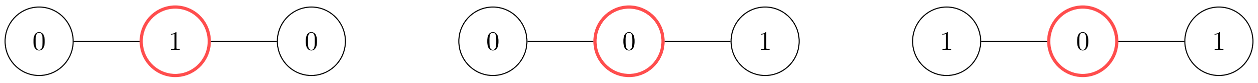


Figure 1: An illustration of the configurations of the MIS problem in cycles. Nodes that are in the MIS output 1 and nodes that are not in the MIS output 0, the center nodes are highlighted in red.

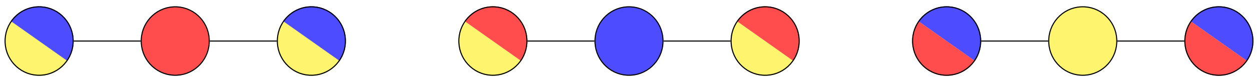


Figure 2: The configurations for the problem of 3 coloring cycles, with colors yellow, blue and red. Two-colored nodes are placeholders, so they represent a node that can either be color 1, or color 2. Each of these 3 configurations represents 4 separate configurations, since the possible combinations of differently colored neighbors have to be turned into explicit versions for the set C.

Task 1: Draw the configurations for 2 coloring cycles and for 2 coloring paths. Note that for paths, we need extra configurations that handle the edgecases of being at the ends of paths.

Complexities of LCLs in Paths and Cycles We now want to establish that LCL problems on paths and cycles have only three possible time complexities:

$$O(1), \qquad \Theta(\log^* n), \qquad \Omega(n).$$

In particular, this implies that *no* LCL on cycles can have a complexity such as $\Theta(\log n)$ or any value different from these three.
To simplify the presentation, we make three assumptions. We can prove the same results also without any of these assumptions, but the details become much more technical.

- We restrict attention to cycles, so we do not need to handle the boundary cases at the ends of a path.
- We assume that the checking radius of Π is 1 (i.e., $r = 1$).
- We assume that there is a consistent orientation of the cycle.
- We use the deterministic LOCAL model.

The results we discuss are due to Yi-Jun Chang, Jan Studený, Jukka Suomela, please refer to their work for the full technical details.

2 Gap between $O(\log^* n)$ and $\Omega(n)$

Our goal is to determine whether a given problem is solvable in $O(\log^* n)$ rounds or whether it necessarily requires $\Omega(n)$ rounds.

High-level plan. Given an LCL problem Π, we construct a graph that represents all allowed sequences of output labels that may appear along a valid solution on an oriented cycle. We call this graph the *sequence graph* of Π. We then define the notion of a *flexible node* in the sequence graph and prove the following dichotomy:

1. If the sequence graph of Π contains a flexible node, then Π can be solved in $O(\log^* n)$ rounds in oriented cycles.
2. If the sequence graph of Π contains no flexible node, then Π requires $\Omega(n)$ rounds in oriented cycles.

Since one of these two cases always holds, every LCL on paths and cycles is either solvable in $O(\log^* n)$ time or requires $\Omega(n)$ time.

The sequence graph. Let $\Pi = (\Sigma, 1, C)$ be an LCL on cycles. Each configuration in C consists of a triple of labeled nodes (v_l, v_c, v_r), standing for the left, center, and right node, with respective output labels $x_l, x_c, x_r \in \Sigma$.
Think of the edge (v_l, v_c): it is labeled by the tuple (x_l, x_c) in that order. Immediately to its right appears the edge (v_c, v_r), which is labeled (x_c, x_r).
The nodes of the sequence graph correspond exactly to such ordered pairs of labels. For every $(x_1, x_2) \in \Sigma^2$ we create a node in the sequence graph.
Two nodes (x_1, x_2) and (y_1, y_2) in the sequence graph are connected by a directed edge if there exists a configuration in C with

$$x_l = x_1, \qquad x_c = x_2 = y_1, \qquad x_r = y_2.$$

That is, an edge labeled (x_1, x_2) may legally be followed by an edge labeled (y_1, y_2) in some valid local configuration of Π.
This construction captures all globally consistent sequences of edge labels that can appear along a cycle in any valid solution. It is this sequence graph that we will analyze to determine the complexity of Π. For example, we have the sequence graph of MIS in Figure 3

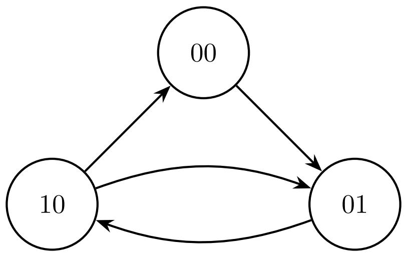


Figure 3: The sequence graph for MIS. Since we now talk about oriented cycles, we need to consider oriented versions of the configuraitons in Figure 1 (the first and third configuration ar symmetric). The first configuraition is $0 \to 1 \to 0$, so this implies an edge from $(0, 1)$ to $(1, 0)$ in the sequence graph. The second configuration is $0 \to 0 \to 1$ this implies an edge from $(0, 0)$ to $(0, 1)$. We can also look at the second configuration from right to left, it then is $1 \to 0 \to 0$ and so it implies an edge from $(1, 0)$ to $(0, 0)$. Lastly the third configuration implies an edge from $(1, 0)$ to $(0, 1)$.

Task 2: Draw the sequence graphs for 3-coloring and 2-coloring.

Walks in the Sequence Graph. A walk in the sequence graph $S_\Pi = (S, E_S)$ is a sequence of output pairs

$$s_1, s_2, s_3, \ldots, s_k \in S,$$

such that for every i, there is a directed edge from s_i to s_{i+1} in S_Π. Note that a walk may revisit the same node multiple times.

We now relate such walks to correct labelings of cycles.
Let $G = (V, E)$ be an oriented cycle, and let $\varphi : V \to \Sigma$ be any correct output labeling for the LCL Π. Consider two consecutive pairs of nodes: u_1, v_1 with v_1 following u_1, and u_2, v_2 with v_2 following u_2. Their output pairs,

$$(x_1, y_1) = (\varphi(u_1), \varphi(v_1)) \quad \text{and} \quad (x_2, y_2) = (\varphi(u_2), \varphi(v_2)),$$

correspond to nodes of the sequence graph S_Π.

Task 3: Assume that the directed distance from u_1 to u_2 along the cycle is k. Show that the labels along the edges between u_1 and u_2 induce a walk in the sequence graph S_Π of length k that starts at (x_1, y_1) and ends at (x_2, y_2).

Flexible Nodes. We call a node (x_1, x_2) of a sequence graph S *flexible* if there exists a constant $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ there exists a walk of length k in S that starts and ends at (x_1, x_2).

For example, in the sequence graph of the MIS problem (Figure 3), the node (01) is flexible with $k_0 = 2$. Indeed, the graph contains a 2-cycle between (01) and (10) (namely $(01) \to (10) \to (01)$), and it also contains a 3-cycle $(01) \to (10) \to (00) \to (01)$. Since every integer $k \geq 2$ can be written as a non-negative integer combination

$$k = 2a + 3b,$$

we may obtain a walk of length k by traversing the 2-cycle a times and the 3-cycle b times. Thus (01) is a flexible state.

Task 4: Do the sequence graphs for 3-coloring and 2-coloring contain flexible nodes? If so, identify a flexible node and justify why it is flexible. If not, argue why no node in the sequence graph is flexible.

We will exploit flexible states as follows.
Consider three nodes u, v, w on the cycle, appearing in this order when traversed clockwise. Assume that the distance from u to v and from v to w is exactly k. Suppose further that there exists a pair (x_1, x_2) such that the sequence graph S_Π contains a walk of length k that starts and ends in (x_1, x_2).

We assign the label x_1 to each of the nodes u, v, w and the label x_2 to their immediate successors along the cycle. (So we have (x_1, x_2) at each of the edges after u, v, w)
Because there exists a length-k walk in S_Π from (x_1, x_2) back to (x_1, x_2), we can use the sequence of output pairs along this walk to determine the labels for all nodes between u and v, and likewise for all nodes between v and w. Thus the segments between u, v, and w can be filled in consistently and in parallel by following the labels prescribed by the walk.

2.1 Flexible Nodes Imply $O(\log^* n)$.

We now prove the following claim:

> Let $\Pi = (\Sigma, 1, C)$ be an LCL on cycles whose sequence graph S_Π contains a flexible node. Then Π can be solved deterministically in $O(\log^* n)$ rounds in the LOCAL model.

The key tool will be the notion of *ruling sets*. Informally, a ruling set on a path or cycle is a subset of nodes that split the path/cycle into similarly sized pieces (like in the u, v, w example above).

Exact Ruling Sets One might attempt to define a ruling set as a set $Z \subseteq V$ containing *exactly every k-th node*. However, on paths this is inherently a global problem: determining which node should serve as the "first" node propagates information throughout the graph. In fact, for any fixed k, computing such a set requires $\Omega(n)$ rounds on paths (and on cycles a solution may not even exist if n is not divisible by k).

Task 5: Show that selecting *exactly every k-th node* on a path requires $\Omega(n)$ rounds. (Hint: Reduce from 2-coloring of paths. Show that, for any constant k, a $o(n)$ algorithm for selecting every k-th node would imply an equally fast deterministic algorithm for 2-coloring a path, which is impossible.)

Because of this inherent global difficulty, we must define ruling sets with some *slack*.

(a, b)-Ruling Sets. A subset $Z \subseteq V$ of the nodes of a cycle is called an (a, b)*-ruling set* (with $1 \leq a \leq b$) if the following hold:

- Any two nodes of Z have distance at least a.
- Every node of the graph, that is not in Z, has a node of Z at distance at most b.

Thus, nodes in Z are neither too close nor too far from each other. For constant a and b, such ruling sets can be computed in $O(\log^* n)$ rounds.

Task 6: Show that for a (k, k)-ruling set Z of a path/cycle, each path of non-ruling set nodes, between two ruling set nodes has length at least $k - 1$ and at most $2k$.

Task 7: Give a deterministic $O(\log^* n)$-round algorithm for computing a (k, k)-ruling set in cycles for any fixed constant k. (Hint: First compute a $O(\log^* n)$-coloring of the k-th power of the cycle[1] using Linial's coloring algorithm. Then iterate through the colors and select nodes for the ruling set, so that the spacing constraints are satisfied.)

Task 8: Assume Π has a flexible node (x_1, x_2) with flexibility bound $k_0 \in O(1)$, and suppose you have computed a $(k_0 + 1, k_0 + 1)$-ruling set Z on the oriented cycle. Describe how to construct a valid solution to Π in $O(k_0) = O(1)$ additional rounds.

Together Tasks 7 and 8 imply, that any LCL Π with a flexible node in its sequence graph can be solved in $O(\log^* n)$ rounds on oriented cycles.

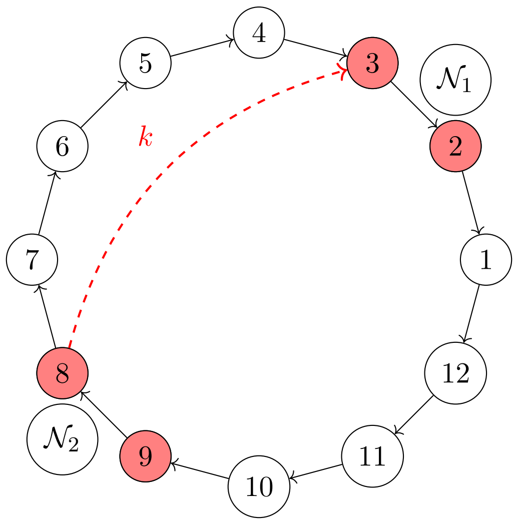
[1] Power graphs should be known from graph theory, but the wikipedia article provides a good reminder.

2.2 No Flexible Node Implies $\Omega(n)$

Next, we show that if the sequence graph S_Π of an LCL Π on oriented cycles contains no flexible node, then Π requires $\Omega(n)$ rounds in the deterministic LOCAL model.

High-Level Idea: If (x_1, x_2) is not a flexible node in the sequence graph, then there are infinitely many integers k for which no walk of length k in the sequence graph starts and ends at (x_1, x_2). For each of these k, this implies that in a solution to Π, there cannot be two pairs of adjacent nodes that output (x_1, x_2) at distance k to each other. (As otherwise by Task 3 there would exist a walk.) To ensure this global constraint, an algorithm must inspect nearly the whole cycle, which forces a lower bound of $\Omega(n)$ rounds.

Illustration. The figure below depicts the idea. Two neighborhoods $\mathcal{N}_1$ (nodes 1–4) and $\mathcal{N}_2$ (nodes 7–10) are shown on the cycle. The algorithm assigns the same label pair (x_1, x_2) to the middle nodes of both neighborhoods (these middle nodes are colored red). The dashed red arrow marks the distance k between the two middle nodes. Since the sequence graph S_Π has no walk of length k that starts and ends at (x_1, x_2), any labeling that places (x_1, x_2) on both middle nodes is invalid, contradicting correctness.



We now make this formal.

Setup. Suppose, for contradiction, that there exists a deterministic LOCAL algorithm A for Π that runs in $o(n)$ rounds on oriented cycles. Since the algorithm has sublinear radius, the output of a pair of consecutive nodes depends only on a neighborhood of radius $o(n)$. Pick n_0 large enough, such that A runs in at most $\frac{n}{C}$ rounds, for all $n \geq n_0$ and some large constant C to be fixed later.

Step 1: Pigeonhole Argument: We need a formal argument, that two such neighborhoods $\mathcal{N}_1$ and $\mathcal{N}_2$ as depicted above must exist.

Task 9: Choose C appropriately (as a function of $|\Sigma|$, the size of the ouptputset), so that there exists an output pair (x_1, x_2) and at least two different $\frac{n}{C}$-hop neighborhoods $\mathcal{N}_1$ and $\mathcal{N}_2$, satisfying the following:

- The set of nodes of the two neighborhoods are assigned disjoint sets of ids.
- In both neighborhoods, there exist two adjacent nodes u, v (v is the successor of u) in the middle, such that A outputs x_1 on u and x_2 on v.

Step 2: No Flexible Node. Take the (x_1, x_2), from step 1, since no node of S_Π is flexible, neither is (x_1, x_2). This means that there exist infinitely many values k such that no walk of length k exists in S_Π starting and ending at (x_1, x_2). We say that such a k is *bad for* (x_1, x_2).

We want to position the two neighborhoods so that their middle nodes are at distance exactly k. However, the radius of each neighborhood depends on the runtime of the algorithm, and the runtime itself depends on the number of nodes in the cycle. Thus, the value of k cannot be chosen arbitrarily: if k is too small, the middle nodes of the two neighborhoods could see each other and the algorithm could not be forced to choose the output (x_1, x_2); if k is too large, we would not have enough nodes in the cycle to place the neighborhoods at distance k. Our construction therefore requires selecting n sufficiently large so that we can choose a value of k that is compatible with both constraints.

Task 10: Use the fact that S_Π does not depend on n and is therefore of finite size. This means *bad* values of k must repeat often.
Then show that starting at some large enough $n_1 \in \mathbb{N}$, for any $n \geq n_1$ and any node (x_1, x_2) of S_Π, there exists a k thats bad for (x_1, x_2) and such that $\frac{n}{6} < k < \frac{n}{3}$.

Step 3: Construct a Cycle Instance. We pick $n > \max\{n_0, n_1\}$ and use Task 9 to obtain two $\frac{n}{C}$-hop neighborhoods $\mathcal{N}_1$ and $\mathcal{N}_2$, such that A outputs the same output (x_1, x_2) on the middle nodes in both neighborhoods. We now place these in on an oriented cycle, such that the middle two nodes of both neighborhoods are exactly distance k apart (where k is the value from Task 10).

Task 11: Argue that when running A on the above instance, A does not produce a correct solution.

Step 4: Conclude the Lower Bound. This shows that algorithm A is not always correct. Since the argument applies to any sublinear-radius algorithm, we conclude:

Any deterministic LOCAL algorithm for Π requires $\Omega(n)$ rounds.