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Dept. of Computer Science
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Theoretical Computer Science - Bridging Course
Exercise Sheet 1

Due: Tuesday, 21st of October 2025, 12:00 pm

Exercise 1: Miscellaneous Mathematical Proofs *(7 Points)*

(a) Let $S(n) = \sum_{i=1}^{n} i$ be the sum of the first n natural numbers and $C(n) = \sum_{i=1}^{n} i^3$ be the sum of the first n cubes. Use mathematical induction to prove the following interesting conclusion: $C(n) = S^2(n)$ for every integer $n \geq 0$. *(3 Points)*

(b) Let A, B, and C be subsets of some nonempty universal set U. Which of the following statements is true? Justify.

- If $A \cap B = A \cap C$, then $B = C$. *(1 Point)*
- If $A \cup B = A \cup C$, then $B = C$. *(1 Point)*
- $\overline{A \cup B} = \overline{A} \cap \overline{B}$, where $\overline{A}$ is the compliment of A. *(2 Points)*

Exercise 2: An Even Degree Node *(3 Points)*

A *simple graph* is a graph without multi-edges (between two nodes there can exist at most one edge) and without self loops (every edge of the graph is an edge between two distinct nodes). Let $G = (V, E)$ be an undirected simple graph. Recall that the degree $d(v)$ of a node $v \in V$ is the number of its neighbors in G , i.e, $d(v) = |\{u \in V \mid \{v, u\} \in E\}|$.

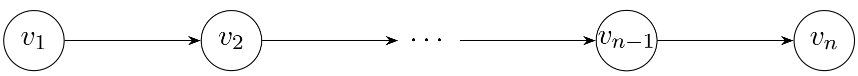
Show that every simple graph with an odd number of nodes contains a node with even degree.

Exercise 3: Visiting All Nodes *(6 Points)*

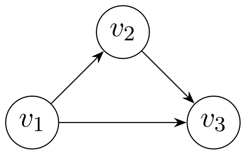
A *complete graph* is a simple undirected graph in which every pair of distinct nodes is connected by a unique edge e.g. a triangle on 3 nodes.

(a) Show that every complete graph G has a path P that visits all the nodes of G. *(1 Point)*

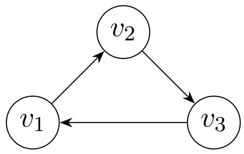
A *directed path* P on n vertices is a simple directed graph whose edge set is the following set of ordered pairs $\{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1 \text{ and } v_i \text{ is a node in } P\}$ i.e. a path in which all the arrows point in the same direction as its steps . We write $P = v_1 v_2 ... v_n$ to denote the directed path P, where below is a visual example.



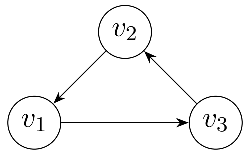
A *tournament* is a complete graph that is oriented, or equivalently a directed graph in which every pair of distinct vertices is connected by a directed edge with any one of the two possible orientations. Below is visual example of 3 different tournements on 3 nodes.



Tournament 1



Tournament 2



Tournament 3

(b) Prove that every tournament T has a directed path P that visits all the nodes of T. *(5 Points)*

Hint: Prove by contradiction. Consider a longest directed path in T and suppose that this path doesn't visit all nodes in T. What happens then?

Exercise 4: Counting Edges in Acyclic Graphs *(4 Points)*

A graph $G = (V, E)$ is *connected* if for all $u, v \in V$, there exists a path from u to v in G. An *acyclic* graph is a graph that contains no cycles. A *tree* is an acyclic, connected, and simple graph.

Show by induction that a tree with $n \geq 1$ nodes has $n-1$ edges.