



Theoretical Computer Science - Bridging Course

Exercise Sheet 10

Due: Tuesday, 13th of January 2026, 12:00 pm

Exercise 1: Propositional Logic: Basic Terms (1+1+1+1 Points)

Let $\Sigma := \{p, q, r\}$ be a set of atoms. An interpretation $I : \Sigma \rightarrow \{T, F\}$ maps every atom to either true or false. Inductively, an interpretation I can be extended to composite formulae φ over Σ (cf. lecture). We write $I \models \varphi$ if φ evaluates to T (true) under I . In case $I \models \varphi$, I is called a *model* for φ .

For each of the following formulae, give *all* interpretations which are models. Make a truth table and/or use logical equivalencies to find all models (document your steps). Which of these formulae are satisfiable, which are unsatisfiable and which are tautologies?

(a) $\varphi_1 = (p \wedge \neg q) \vee (\neg p \vee q)$

(b) $\varphi_2 = (\neg p \wedge (\neg p \vee q)) \leftrightarrow (p \vee \neg q)$

(c) $\varphi_3 = (p \wedge \neg q) \rightarrow \neg(p \wedge q)$

(d) $\varphi_4 = (p \wedge q) \rightarrow (p \vee r)$

Remark: $a \rightarrow b \equiv \neg a \vee b$, $a \leftrightarrow b \equiv (a \rightarrow b) \wedge (b \rightarrow a)$, $a \not\rightarrow b \equiv \neg(a \rightarrow b)$.

Exercise 2: CNF and DNF (2+2 Points)

(a) Convert $\varphi_1 := (p \rightarrow q) \rightarrow (\neg r \wedge q)$ into Conjunctive Normal Form (CNF).

(b) Convert $\varphi_2 := \neg((\neg p \rightarrow \neg q) \wedge \neg r)$ into Disjunctive Normal Form (DNF).

Remark: Use the known logical equivalencies given in the lecture slides to do the necessary transformations. State which equivalency you are using in each step.

Exercise 3: Logical Entailment (3+3 Points)

A *knowledge base* KB is a set of formulae over a given set of atoms Σ . An interpretation I of Σ is called a *model* of KB , if it is a model for *all* formulae in KB . A knowledge base KB *entails* a formula φ (we write $KB \models \varphi$), if *all* models of KB are also models of φ .

Let $KB := \{p \vee q, \neg r \vee p\}$. Show or disprove that KB logically entails the following formulae.

(a) $\varphi_1 := (p \wedge q) \vee \neg(\neg r \vee p)$

(b) $\varphi_2 := (q \leftrightarrow r) \rightarrow p$

Exercise 4: Inference Rules and Calculi

(3+3 Points)

Let $\varphi_1, \dots, \varphi_n, \psi$ be propositional formulae. An *inference rule*

$$\frac{\varphi_1, \dots, \varphi_n}{\psi}$$

means that if $\varphi_1, \dots, \varphi_n$ are 'considered true', then ψ is 'considered true' as well ($n = 0$ is the special case of an axiom). A (propositional) *calculus* **C** is described by a *set* of inference rules.

Given a formula ψ and knowledge base $KB := \{\varphi_1, \dots, \varphi_n\}$ (where $\varphi_1, \dots, \varphi_n$ are formulae) we write $KB \vdash_{\mathbf{C}} \psi$ if ψ can be derived from KB by starting from a subset of KB and repeatedly applying inference rules from the calculus **C** to 'generate' new formulae until ψ is obtained.

Consider the following two calculi, defined by their inference rules (φ, ψ, χ are arbitrary formulae).

$$\begin{aligned} \mathbf{C}_1 : & \quad \frac{\varphi \rightarrow \psi, \psi \rightarrow \chi}{\varphi \rightarrow \chi}, \frac{\neg\varphi \rightarrow \psi}{\neg\psi \rightarrow \varphi}, \frac{\varphi \leftrightarrow \psi}{\varphi \rightarrow \psi, \psi \rightarrow \varphi} \\ \mathbf{C}_2 : & \quad \frac{\varphi, \varphi \rightarrow \psi}{\psi}, \frac{\varphi \wedge \psi}{\varphi, \psi}, \frac{(\varphi \wedge \psi) \rightarrow \chi}{\varphi \rightarrow (\psi \rightarrow \chi)} \end{aligned}$$

Using the respective calculus, show the following derivations (document your steps).

- (a) $\{p \leftrightarrow \neg r, \neg q \rightarrow r\} \vdash_{\mathbf{C}_1} p \rightarrow q$
- (b) $\{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow s\} \vdash_{\mathbf{C}_2} s$

Remark: Inferences of a given calculus are purely syntactical, i.e. rules only apply in their specific form (much like a grammar) and no other logical transformations not given in the calculus are allowed.

Exercise 5: Resolution Calculus

(Bonus Points)

Due to the *Contradiction Theorem* (cf. lecture) for every knowledge base KB and formula φ it holds

$$KB \models \varphi \iff KB \cup \{\neg\varphi\} \models \perp.$$

Remark: \perp is a formula that is unsatisfiable.

In order to show that KB entails φ , we show that $KB \cup \{\neg\varphi\}$ entails a contradiction. A calculus **C** is called *refutation-complete* if for every knowledge base KB

$$KB \models \perp \implies KB \vdash_{\mathbf{C}} \perp.$$

Hence, given a refutation-complete calculus **C** it suffices to show $KB \cup \{\neg\varphi\} \vdash_{\mathbf{C}} \perp$ to prove $KB \models \varphi$.

The *Resolution Calculus* **R** is a formal way to do a prove by contradiction. It is correct and refutation-complete¹ for knowledge bases that are given in *Conjunctive Normal Form* (CNF). A knowledge base KB is in CNF if it is of the form $KB = \{C_1, \dots, C_n\}$ where its clauses $C_i = \{L_{i,1}, \dots, L_{i,m_i}\}$ each consist of m_i literals $L_{i,j}$.

Remark: KB represents the formula $C_1 \wedge \dots \wedge C_n$ with $C_i = L_{i,1} \vee \dots \vee L_{i,m_i}$.

The Resolution Calculus has only one inference rule, the *resolution rule*:

$$\mathbf{R} : \quad \frac{C_1 \cup \{L\}, C_2 \cup \{\neg L\}}{C_1 \cup C_2}.$$

Remark: L is a literal and $C_1 \cup \{L\}, C_2 \cup \{\neg L\}$ are clauses in KB (C_1, C_2 may be empty). To show $KB \vdash_{\mathbf{R}} \perp$, you need to apply the resolution rule, until you obtain two conflicting one-literal clauses L and $\neg L$. These entail the empty clause (defined as \square), i.e. a contradiction ($\{L\}, \{\neg L\} \vdash_{\mathbf{R}} \perp$).

¹Complete calculi are impractical, since they have too many inference rules. More inference rules make automated proving with a computer significantly more complex. The Resolution Calculus is an appropriate technique to avoid this additional complexity, since it has only one inference rule.

- (a) We want to show $\{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow u\} \models u$. First convert this problem instance into a form that can be solved via resolution as described above. Document your steps.
- (b) Now, use resolution to show $\{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow u\} \models u$.
- (c) Consider the sentence “Heads, I win”. “Tails, you lose”. Design a propositional KB that represents these sentences (create the propositions and rules required). Then use propositional resolution to prove that **I always win**.