



Algorithms and Data Structures – Conditional Course Winter Term 2026/27

Exercise Sheet 13

Due: Wednesday, 03 February 2027, 12:00

Exercise 1: Minimum Spanning Trees (5 Bonus Points)

Let $G = (V, E)$ be a connected undirected graph with pairwise distinct edge weights, and let T be an MST of G .

- (a) Let e be a minimum-weight edge of G . Show that some MST of G contains e . (1 Point)
- (b) Let e be a maximum-weight edge on a cycle C in G . Show that no MST of G contains e . (3 Points)
- (c) Briefly explain which statement, (a) or (b), supports the correctness of Kruskal's algorithm. (1 Point)

Exercise 2: Insertion Sort and Inversions (7 Bonus Points)

Let $A[1, \dots, n]$ be an array of pairwise distinct numbers. An *inversion* is a pair of indices (i, j) satisfying

$$i < j \quad \text{and} \quad A[i] > A[j].$$

Let $I(A)$ be the number of inversions. Consider the version of Insertion Sort that repeatedly swaps an element with its left neighbor until it reaches its correct position.

- (a) Show that swapping two adjacent elements that form an inversion reduces the number of inversions by exactly one. (2 Points)
- (b) Show that every swap performed by Insertion Sort exchanges two elements that form an inversion. (2 Points)
- (c) Conclude that Insertion Sort performs exactly $I(A)$ swaps. (1 Point)
- (d) Give an asymptotic bound on its running time in terms of n and $I(A)$. Briefly justify your answer. (2 Points)

Exercise 3: Asymptotic Growth

(6 Points)

Assume that all logarithms have base 2.

- (a) Order the following functions by asymptotic growth. Explicitly indicate any functions that grow at the same asymptotic rate. No proofs are required.

$$\begin{aligned} f_1(n) &= n^{3/2}, & f_2(n) &= 2^{\log n + \sqrt{\log n}}, & f_3(n) &= (\log n)^{10}, \\ f_4(n) &= n \log n, & f_5(n) &= 4^{\log n}. \end{aligned}$$

(3 Points)

- (b) Prove or disprove, using the definition of Landau notation:

$$n(\log n)^2 \in O\left(n^{4/3}\right).$$

(3 Points)

Exercise 4: Hashing

(7 Points)

Store data in a hash table of size $m = 11$. Resolve collisions by open addressing with double hashing. The probing function is

$$h(x, j) = (h_1(x) + j \cdot h_2(x)) \bmod 11,$$

where

$$h_1(x) = x \bmod 11 \quad \text{and} \quad h_2(x) = 1 + (x \bmod 10).$$

Insert the keys 22, 41, 53, 46, 30, 13, in this order, into an initially empty table.

- (a) Insert all keys. For each key, list every position probed before insertion. (3 Points)
- (b) Give the final contents of the table. (1 Point)
- (c) Explain why each key's probing sequence visits every table position before repeating a position. (2 Points)
- (d) What is the worst-case running time of an unsuccessful search? Give an asymptotic answer for a general table size m . (1 Point)